RADEMACHER'S THEOREM IN METRIC MEASURE SPACES AND ALBERTI REPRESENTATIONS

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1. Introduction

Rademacher's theorem states that any Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable Lebesgue almost everywhere. It is a fundamental result in geometric measure theory. For example, it shows that any rectifiable set possesses a weak tangent plane at almost every point. Moreover the statement itself is very interesting in its own right; the fact that a seemingly rather simple condition can impose such strong regularity is quite remarkable. Rademacher's theorem leads to many more questions that can be answered.

Recently there has been a wealth of interest in generalising results of classical analysis to the setting of metric spaces, possibly with an underlying measure. Naturally, Rademacher's theorem is a candidate for such a generalisation. This course will focus on a new proof of Cheeger's generalisation which replaces the domain with a doubling metric measure space (X, d, μ) that satisfies a Poincaré inequality [Che99]. This proof uses modern techniques developed in [Bat14] which consider a rich structure of Lipschitz curves in the metric space (known as "Alberti representations") which allow us to form a partial derivative of any Lipschitz function $f\colon X \to \mathbb{R}$. By considering many such families of curves, we are able to form a derivative of f and hence deduce Cheeger's theorem.

This constitutes the third proof of Cheeger's theorem. In addition to the original, Keith [Kei04] gave an independent proof (see also [KM16]). One common element to all three of these proofs is the use of weak tangent spaces (defined via Gromov-Hausdorff convergence), which is only natural considering the nature of the derivative. Essentially, after an analysis of a large collection of Lipschitz functions in the metric space X, one takes a weak limit and the tangential behaviour of these Lipschitz functions is significantly more rigid (i.e. linear in the classical theory). The doubling condition then gives an upper bound on the number of different (i.e. linearly independent) tangential behaviours the large set of Lipschitz functions can posses, which ultimately leads to the existence of a derivative.

Throughout, a *metric measure space* will refer to a triple (X, d, μ) consisting of a complete metric space (X, d) equipped with a σ-finite Borel measure μ.

For many purposes, this definition includes spaces that are simply too large to work in. Often, it is natural to impose the following mild "finite dimensional" condition.

Given a metric space, unless otherwise stated, we will denote by $B(x, r)$ the closed ball centred at x with radius $r > 0$. Given a ball B and $\lambda > 0$, we will denote by λB the ball with the same centre and with radius λ times bigger.

A metric measure space (X, d, μ) is *doubling* if there exists a $C_D \geq 1$ (the *doubling constant* of μ) such that, for each ball $B \subset X$,

$$0 < \mu(2B) \leq C_D\mu(B) < \infty.$$

In particular, by induction,

$$\mu(2^n B) \leq C_D^n \mu(B)$$

for each $n \in \mathbb{N}$.

A standard property of doubling measures is that they satisfy the Vitali covering theorem and hence the Lebesgue differentiation and density theorems. The proofs of these are exactly the same as for Lebesgue measure.

Doubling metric measure spaces are also doubling metric spaces. That is, there exists an $N_D \in \mathbb{N}$ such that any ball B is covered by at most N_D balls of half the radius of B. Indeed, if $\mathcal{B}$ is a maximal disjoint collection of balls centred in B with radius 1/4 the radius of B, then $\mathcal{B}$ can contain at most $C_D^3 =: N_D$ elements $B_1, \ldots, B_{N_D}$. Since $\mathcal{B}$ is maximal

$$B \subset 2B_1 \cup 2B_2 \cup \ldots \cup 2B_{N_D}.$$

Recall that a function $f\colon (X,d) \to (Y,\rho)$ is *Lipschitz* if there exists an $L \geq 0$ such that

$$\rho(f(x), f(y)) \leq L d(x,y)$$

for each $x, y \in X$. The smallest such L is called the *Lipschitz constant* of f.

Theorem 1.1 (Lebesgue $n = 1$, Rademacher $n > 1$)**.** *Any Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere.*

We also require the fundamental theorem of calculus.

Theorem 1.2 (Fundamental theorem of calculus)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then for any $x \leq y \in \mathbb{R}$*

$$f(y) - f(x) = \int_x^y Df \, \mathrm{d}\mathcal{L}.$$

2. The Poincaré inequality in metric measure spaces and Cheeger's theorem

The proof of the classical Poincaré inequality relies on the fundamental theorem of calculus. This motivates our definition of a Poincaré inequality in the setting of metric measure spaces, but we must replace a "line" by a rectifiable curve.

Given a Lipschitz $\gamma\colon [a,b] \to X$ there exists a re-parametrisation $\gamma^*\colon [0,l] \to X$ of γ called its *arc-length* parametrisation. This is a 1-Lipschitz function with the property that, if γ is injective, so is γ^* and $\mathcal{H}^1(\gamma^*(S)) = \mathcal{L}^1(S)$ for every measurable $S \subset [0,l]$. This is the natural parametrisation for integrating on the image of γ.

Definition 2.1. Given a Borel function $\rho\colon X \to \mathbb{R}$ we define the *line integral* of ρ over a rectifiable curve γ to be

$$\int_\gamma \rho \, \mathrm{d}s = \int_0^l \rho \circ \gamma^*(t) \, \mathrm{d}t.$$

If γ is injective, this agrees with

$$\int_{\gamma([a,b])} \rho \, \mathrm{d}\mathcal{H}^1.$$

This allows us to define a replacement for the fundamental theorem of calculus in a metric space.

Definition 2.2. Let X be a metric space and $f\colon X \to \mathbb{R}$ a Lipschitz function. A Borel function $\rho\colon X \to [0,\infty)$ is an *upper gradient* of f if, for every $x, y \in X$ and every rectifiable curve γ joining x to y,

$$|f(x) - f(y)| \leq \int_\gamma \rho \, \mathrm{d}s.$$

Observe that, if $X = \mathbb{R}^n$, then by Lebesgue's theorem, $\rho = \|Df\|$ is an upper gradient of any Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$. More generally, for any metric space X and any Lipschitz $f\colon X \to \mathbb{R}$, $\operatorname{Lip} f$ is an upper gradient of f, as is the local Lipschitz constant $\operatorname{Lip}(f, \cdot)$.

Observe also that this definition is only of value when X has a rich structure of rectifiable curves. For example, if X contains no non-trivial rectifiable curves, as is the case when $X = \{0, 1\}$, then $\rho = 0$ is an upper-gradient of *any* Lipschitz function.

The notion of a Poincaré inequality in a metric measure space says that any upper gradient of a Lipschitz function must control the behaviour in a very precise way. The Poincaré inequality in this setting was first introduced by Heinonen and Koskela [**Heinonen˙1998**]. We give a slightly different formulation that is equivalent whenever the measure is doubling.

Definition 2.3. For $p \geq 1$, a metric measure space (X, d, μ) satisfies a *p-Poincaré inequality* if there exists a $C \geq 1$ such that, for any ball $B \subset X$, any Lipschitz $f\colon X \to \mathbb{R}$ and any upper-gradient ρ of f,

$$\fint_B |f - f_B| \leq C \operatorname{rad}(B) \left(\fint_B \rho^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

We say that (X, d, μ) is a *p-PI space* if it is doubling and satisfies a p-Poincaré inequality.

By Holder's inequality, the Poincare inequality becomes weaker as p increases. The classical Poincaré inequality shows that Euclidean space is a 1-PI space.

Laakso gave several very interesting, non-trivial, and highly non-Euclidean examples of 1-PI spaces [**laakso**; **laakso-graph**].

We now move on to considering the derivatives of real valued Lipschitz functions defined on a metric space. The first way to generalise a derivative to this setting was introduced by Cheeger [Che99] and was later refined by Keith [Kei04]. It simply replaces to coordinate functions in Euclidean space by an arbitrary vector valued Lipschitz function.

Definition 2.4. Let $\phi\colon X \to \mathbb{R}^n$ be a fixed Lipschitz function and $x_0 \in X$.

A function $f\colon X \to \mathbb{R}$ is *differentiable with respect to* ϕ at x_0 if there exists a unique $Df(x_0) \in \mathbb{R}^n$ such that

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\phi(x) - \phi(x_0))|}{d(x, x_0)} = 0.$$

Equivalently, $\operatorname{Lip}(f - Df(x_0) \cdot \phi, x_0) = 0$.

Note that we must *assume* that the derivative is unique as a part of the definition.

We can now say what it means for a metric measure space to satisfy a generalisation of Rademacher's theorem.

Definition 2.5. A metric measure space (X, d, μ) is a *Lipschitz differentiability space* if there exists a countable Borel decomposition

$$X = \bigcup_{i \in \mathbb{N}} U_i$$

and countably many Lipschitz functions $\phi_i\colon X \to \mathbb{R}^{n_i}$ such that the following is true: For every Lipschitz $f\colon X \to \mathbb{R}$ and every $i \in \mathbb{N}$, f is differentiable at μ-a.e. point in U_i with respect to ϕ_i.

We call the pair (U_i, ϕ_i) a *chart*.

This is all the required concepts to state Cheeger's theorem [Che99].

Theorem 2.6 (Cheeger)**.** *Any PI space is a Lipschitz differentiability space.*

The fact that a function f is differentiable with respect to ϕ at a point x_0 should be compared to the statement that a vector v is a linear combination of a collection of vectors $p_1, p_2, \ldots, p_n$. Essentially, it says that the components of ϕ span the possible infinitesimal behaviour of Lipschitz functions at x_0.

The fact that the derivative is unique is comparable to the fact that the components of ϕ are linearly independent. Indeed, since the zero function must have a unique derivative, namely derivative zero, at almost every point, we see that for almost every x_0, $\mathrm{Lip}(D \cdot \phi, x_0) = 0$ implies $D = 0$.

This idea can be expanded upon to give the following characterisation of Lipschitz differentiability spaces. Note that it comparable to the following statement: Suppose that, for a vector space V, there exists a $N \in \mathbb{N}$ such that any collection of linearly independent vectors has size at most N. Then V has a basis of size at most N.

We require the following measure theoretic property: Suppose that μ is a σ-finite measure on X and $\mathcal{T}$ be a collection of μ measurable sets such that any positive measure subset of X contains a positive measure element of $\mathcal{T}$. Then we can decompose almost all of X into a countable union of elements of $\mathcal{T}$.

Proposition 2.7. *Let (X, d, μ) be a σ-finite metric measure space and suppose that there exists an $N \in \mathbb{N}$ for which the following is true. When ever $\phi \colon X \to \mathbb{R}^n$ is Lipschitz with the property that*

$$\mathrm{Lip}(D \cdot \phi, x) > 0 \quad \text{for every } D \in \mathbb{R}^n \setminus \{0\}, \tag{2.1}$$

for a set of positive measure $x \in X$, then $n \leq N$. Then (X, d, μ) is a Lipschitz differentiability space.

Proof. Let $U \subset X$ be a Borel set of positive measure. Either every Lipschitz $\phi \colon X \to \mathbb{R}$ satisfies $\mathrm{Lip}(\phi, x) = 0$ for μ almost every $x \in U$ or there exists a $U_1 \subset U$ of positive measure and a Lipschitz $\phi_1 \colon X \to \mathbb{R}$ with $\mathrm{Lip}(\phi_1, x) > 0$ for every $x \in U_1$. Given the first option we stop; in this case X is a Lipschitz differentiability space with respect to the zero function. Otherwise we proceed iteratively.

Suppose that we have, for some $n \in \mathbb{N}$, a Lipschitz $\phi_n \colon X \to \mathbb{R}^n$ and a $U_n \subset U$ with $\mu(U_n) > 0$ such that $\mathrm{Lip}(D \cdot \phi, x) > 0$ for every $D \in \mathbb{S}^{n-1}$ for almost every $x \in U_n$. Then either

- (U_n, ϕ) is a chart with respect to which every Lipschitz $f \colon X \to \mathbb{R}$ is differentiable almost everywhere;
- or there exists a $U_{n+1} \subset U_n$ of positive measure and a Lipschitz $\phi^{n+1} \colon X \to \mathbb{R}$ such that $\phi' = (\phi, \phi^{n+1})$ satisfies $\mathrm{Lip}(D \cdot \phi', x) > 0$ for every $D \in \mathbb{R}^{n+1} \setminus \{0\}$.

As we mentioned above, in the first case, the uniqueness of the derivative is equivalent to the induction hypotheses on ϕ.

By hypothesis, the second option cannot hold for sufficiently large n, and so at some point we find a subset of U of positive measure which forms a chart. The standard measure theory result completes the proof. □

3. Alberti representations as an alternative generalisation of Rademacher's theorem

We now introduce an alternative way to generalise Rademacher's theorem to metric measure spaces. However, it turns out that this in an equivalent formulation to Cheeger's. This is nice because it allows us to have two very different descriptions of the same phenomenon.

This alternative formulation is based on the idea of forming a *partial derivative* of any Lipschitz function along a rectifiable curve.

Definition 3.1. Given a metric space X, define the set of *curve fragments* in X to be

$$\Gamma(X) = \{\gamma\colon \operatorname{dom}\gamma \subset [0,1] \to X : \operatorname{dom}\gamma \text{ compact, non-empty, } \gamma \text{ 2-biLipschitz}\}.$$

Given $\gamma \in \Gamma(X)$ let

$$\operatorname{graph}(\gamma) = \{(t,\gamma(t)) : t \in \operatorname{dom}\gamma\} \subset [0,1] \times X.$$

This is injective and so we can define a metric on $\Gamma(X)$ as the Hausdorff metric on the graphs of curve fragments.

We do not assume that $\operatorname{dom}\gamma$ is connected; one should consider the general case that $\operatorname{dom}\gamma$ is a fat Cantor set. This is important, so that $\Gamma(X)$ is much richer than the set of *curves* in X.

The details of this metric are not important for us. The key point is that it allows us to consider Borel measures on Γ.

Definition 3.2. An *Alberti representation* of a metric measure space (X,d,μ) consists of a probability measure $\mathbb{P}$ on $\Gamma(X)$ and for each $\gamma \in \Gamma(X)$ a measure $\mu_\gamma \ll \mathcal{H}^1|_{\gamma([0,1])}$ such that

$$\mu(B) = \int_{\Gamma(X)} \mu_\gamma(B)\, \mathrm{d}\mathbb{P}(\gamma)$$

for every Borel $B \subset X$.

Fubini's theorem shows that Lebesgue measure has many Alberti representations.

The motivation for considering Alberti representations is the following. Suppose that $f\colon X \to \mathbb{R}$ is Lipschitz and that $\gamma \in \Gamma(X)$. Then the composition

$$f \circ \gamma\colon \operatorname{dom}\gamma \to \mathbb{R}$$

is Lipschitz and so is differentiable almost everywhere by Lebesgue's theorem. That is, for $\mathcal{H}^1$ almost every $x \in \operatorname{im}\gamma$, there exists a *partial derivative* of f at x given by $(f\circ\gamma)'(\gamma^{-1}(x))$. It is then not hard to see that, if (X,d,μ) has an Alberti representation, then such a partial derivative exists for μ almost every $x \in X$.

This should be compared to a standard proof of Rademacher's theorem; a first step is to use the Fubini and Lebesgue theorems to find a partial derivative of any Lipschitz function. Here, we are replacing the use of Fubini's theorem with the *definition* of an Alberti representation.

Of course, in the proof of Rademacher's theorem, it is important that we can find all n partial derivatives of a Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$. Our next step is to interpret this for Alberti representations.

In Euclidean space, we can distinguish different families of curves by considering their tangents: some curves travel in a direction close to the e_1 direction, some others close to the e_2 direction and so on. In metric spaces there is no natural way to assign a tangent. However, we can use a Lipschitz $\phi\colon X \to \mathbb{R}^n$ to pull back the geometry of $\mathbb{R}^n$ to the metric space.

Definition 3.3. Let (X,d) be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. We say that $\gamma \in \Gamma(X)$ is a C-curve (with respect to ϕ) if

$$\phi(t) - \phi(s) \in C \setminus \{0\} \quad \text{for every } s,t \in \operatorname{dom}\gamma.$$

Similarly, an Alberti representation $(\mathbb{P},\{\mu_\gamma\})$ of a metric measure space is a C-Alberti representation (with respect to ϕ) if $\mathbb{P}$ almost every curve is a C curve (with respect to ϕ).

Finally, we say that a collection of Alberti representations

$$(\mathbb{P}_1, \{\mu^1_\gamma\}), \ldots, (\mathbb{P}_n, \{\mu^n_\gamma\})$$

is *independent* if there exists a Lipschitz $\phi\colon X \to \mathbb{R}^n$ and independent cones $C_1, \ldots, C_n \subset \mathbb{R}^n$ such that $(\mathbb{P}_i, \{\mu^i_\gamma\})$ is a C_i-Alberti representation with respect to ϕ for each $1 \le i \le n$.

We say that cones $C_1, \ldots, C_n \subset \mathbb{R}^n$ are *independent* if any choice $v_i \in C_i \setminus \{0\}$ forms a linearly independent set.

Often the underlying function ϕ will be clear from the context and we will not explicitly mention it.

We see from Fubini's theorem that Lebesgue measure has a collection of n independent Alberti representations (and no more), and that this is precisely the property we required in our proof of Rademacher's theorem.

The first half of our proof of Cheeger's theorem will be to prove that a PI space has a large collection of independent Alberti representations. Before we prove this, we will develop a general technique for deciding when a measure has many independent Alberti representations.

Definition 3.4. Let X be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. A Borel $S \subset X$ is *C-null (with respect to ϕ)* if $\mathcal{H}^1(\gamma \cap S) = 0$ for each C-curve γ.

Observe that, if a metric measure space is C-null then it cannot support a C-Alberti representation. This is in fact a characterisation of those measures with C-Alberti representations. We will use this fact without proof.

Proposition 3.5. *Let (X, d, μ) be a metric measure space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. There exists a decomposition $X = A \cup S$ such that $\mu \llcorner A$ has a C-Alberti representation and S is C-null. This decomposition is unique up to μ null sets.*

Compare to the Lebesgue decomposition theorem.

By applying this lemma to a collection of independent cones $C_1, C_2, \ldots, C_n$, we obtain a decomposition $X = A \cup S_1 \cup S_2 \cup \ldots \cup S_n$ where each S_i is C_i null. and $\mu|_A$ has n independent Alberti representations. This is a good start, but, in a PI space, we will only be able to prove that the S_i have measure zero when θ is sufficiently wide. Of course, we cannot simply apply the lemma to very wide cones, because they will not be independent. Therefore, we must do something more involved. The first step is to show that we can combine the cones of C-null sets.

Lemma 3.6. *Let X be a metric space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. Suppose that $S \subset X$ is C-null with respect to ϕ. Then for any $\gamma \in \Gamma(X)$,*

$$(\phi \circ \gamma)'(t) \notin \operatorname{int}(C)$$

for almost every $t \in \gamma^{-1}(S)$.

Proof. Suppose that the conclusion is false and that for some $\gamma \in \Gamma(X)$,

$$B = \{t \in \gamma^{-1}(S) : (\phi \circ \gamma)'(t) \in \operatorname{int}(C)\}$$

has positive measure. Let

$$\operatorname{int}(C) = \bigcup_{i \in \mathbb{N}} C_i$$

where each C_i is closed and convex. Then there exists some $i \in \mathbb{N}$ such that

$$B_1 = \{t \in B : (\phi \circ \gamma)'(t) \in C_i\}$$

has positive measure. Further, since C_i is a closed subset of $\text{int}(C)$, there exists an $R > 0$ such that the set

$$B_2 = \{t \in B_1 : \frac{\phi(\gamma(t+r)) - \phi(\gamma(t))}{r} \in C \setminus \{0\} \ \forall\ 0 < |r| < R\}$$

also has positive measure.

By dividing B_2 into finitely many subsets of diameter R, we can find one such subset B_3 of positive measure. In particular, for any $s, t \in B_3$,

$$\frac{\phi(\gamma(t)) - \phi(\gamma(s))}{t - s} \in C \setminus \{0\}.$$

Finally, by taking a compact $K \subset B_3$ of positive measure, we see that $\gamma|_K$ is a C-curve intersecting S in a set of positive measure. This is a contradiction. □

The previous lemma gives us the following method to "refine" the directions of an Alberti representation. Of course, this makes most sense when the C_i are very thin.

Lemma 3.7. *Let X be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. Suppose that a measure μ has a C-Alberti representation with respect to ϕ. Then, for any collection of cones $C_1, \ldots, C_m \subset \mathbb{R}^n$ such that*

$$C \setminus \{0\} \subset \bigcup_{i=1}^{m} \text{int}(C_i),$$

there exists a Borel decomposition $X = X_1 \cup \ldots \cup X_m$ such that each $\mu|_{X_i}$ has a C_i-Alberti representation.

Proof. By Proposition 3.5 there exists a decomposition $X = X_1 \cup S_1$ such that X_1 has the required form and S_1 is C_1-null. By applying Proposition 3.5 again (to $\mu|_{S_1}$) we obtain a decomposition $X = X_1 \cup X_2 \cup S_2$ where X_2 has the required form and S_2 is both C_1-null and C_2-null. By repeating, we obtain a decomposition $X = X_1 \cup \ldots \cup X_m \cup S$ such that each X_i has the required form and S is C_i-null for each $1 \leq i \leq m$. Therefore, by Lemma 3.6, S is C-null. Since μ has a C-Alberti representation, we must have $\mu(S) = 0$. Therefore, $X_1' = X_1 \cup S$ also has the required properties of X_1, and $X = X_1' \cup X_2 \cup \ldots \cup X_m$ is the required decomposition. □

Finally we can prove the decomposition that we are after.

Definition 3.8. For $w \in \mathbb{S}^{n-1}$ and $0 < \theta < 90$, let $C(w, \theta) \subset \mathbb{R}^n$ be the cone centred on w with interior angle θ.

Let X be a metric space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. Define the set $\tilde{A}(\phi)$ to be the collection of $S \subset X$ for which the following is true: For any $0 < \theta < 1$ there exists a Borel decomposition

$$S = S_1 \cup S_2 \cup \ldots \cup S_m$$

and cones $C_1, C_2, \ldots C_m$ of interior angle $90 - \theta$ such that S_i is C_i-null for each $1 \leq i \leq m$.

Theorem 3.9. *Let (X, d, μ) be a metric measure space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. There exists a Borel decomposition*

$$X = S \cup \bigcup_{i \in \mathbb{N}} A_i$$

such that each $\mu|_{A_i}$ has n independent Alberti representations and $S \in \tilde{A}(\phi)$.

Proof. We first prove the following. Given any $0 < \theta < 1$ there exists a Borel decomposition

$$X = \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{m} S_i \tag{3.1}$$

such that each $\mu|_{A_i}$ has n independent Alberti representations and each S_i is C_i-null, for some C_i with interior angle $90-\theta$. We will iteratively find theese Alberti representations one-by-one.

First we apply Proposition 3.5 to an arbitrary cone C of interior angle $90-\theta$ to obtain a decomposition $X = A \cup S$ where $\mu|_A$ has a C-Alberti representation and S is C-null.

Now suppose that, for $1 \leq d < n$ we have a decomposition as in eq. (3.1) where each $\mu|_{A_i}$ has d independent Alberti representations. For a $0 < \alpha < 1$ to be determined later, by applying Lemma 3.7 (and increasing m) we may suppose that there exists $w_1^i, \ldots, w_d^i \in \mathbb{S}^{n-1}$ such that these Alberti representations are $C(w_j^i, \alpha)$-Alberti representations.

Fix a $1 \leq i \leq m$ and pick w_{d+1}^i to be orthogonal to $w_1^i, \ldots, w_d^i$, which is possible because, by assumption, $1 \leq d < n$. We pick α so small that $C(w_{d+1}^i, 90-\theta)$ is disjoint from each $C(w_j^i, \alpha)$ (this is independent of the choice of the w_j^i and only depends on n). Then the cones $C(w_1^i, \alpha), \ldots, C(w_d^i, \alpha)$ and $C(w_{d+1}^i, \theta)$ are independent.

Finally, we apply Proposition 3.5 to obtain a decomposition $A_i = A_i' \cup S'$ where $\mu|_{A_i'}$ has $d+1$ independent Alberti representations and S' is $C(w_{d+1}^i, 90-\theta)$-null.

Repeating this for each $1 \leq d < n$ completes the proof of eq. (3.1). To complete the proof of the theorem we apply eq. (3.1) for each $j \in \mathbb{N}$ with $\theta = 1/j$ to obtain a decomposition $X = \hat{A}_j \cup \hat{S}_j$ where each $\hat{A}_j$ is a finite union of sets with n independent Alberti representations and each $\hat{S}_j$ has a finite decomposition into sets that are C-null for some cone C of interior angle $90-1/j$. Setting $S = \cap_j \hat{S}_j$ completes the proof. □

4. Alberti representations and the Poincaré inequality

The next step is to demonstrate that PI spaces possess many independent Alberti representations. Of course, we cannot use Theorem 3.9 to deduce that there arbitrarily large collections of independent Alberti representations; $\mathbb{R}^n$ can have at most n independent Alberti representations. Thus we must find some reasonable conditions on ϕ for which that proposition is useful (see also the exercises for when it is vacuous).

This is because, for poorly chosen ϕ, the whole of X is a $\tilde{A}(\phi)$ set. For example, consider what happens when $\phi = 0$, or $X = \mathbb{R}$ and $\phi(x) = (x, x)$. The problem is that the components of ϕ are linearly dependent. This is precisely where the condition eq. (2.1) is used.

Lemma 4.1. *Let (X, d, μ) be a doubling metric measure space and $f\colon X \to \mathbb{R}$ 1-Lipschitz. There exists a constant $\eta = \eta(\mathrm{doub}(\mu)) \geq 1$ such that, for any ball $B \subset X$,*

$$\fint_{2B} |f - f_{2B}| \,\mathrm{d}\mu \geq C(\mu) \operatorname{rad} B \sup\left\{ \frac{|f(x) - f(y)|}{\operatorname{rad} B} : x, y \in B \right\}^{\eta}.$$

In particular, if (X, d, μ) is a PI space and ρ is an upper gradient of f,

$$\mathrm{Lip}(f, x) \leq C(\mathrm{PI}(\mu)) \rho(x)^{1/\eta}$$

for μ almost every $x \in X$.

Remark 4.2. The 'in particular' statement of the previous lemma is the only part in the proof that we require the Poincaré inequality.

Proof. Fix a ball $B \subset X$ and let

$$M = \sup\left\{\frac{|f(x)-f(y)|}{\operatorname{rad} B} : x, y \in B\right\} \leq 1$$

If $M = 0$ there is nothing to prove and so we may suppose $M > 0$. Since B is compact, there exist $x_1, x_2 \in B$ such that

$$|f(x_1) - f(y_2)| = M \operatorname{rad} B.$$

For $i = 1, 2$ let $B_i = B(x_i, M \operatorname{rad} B/10) \subset 2B$. Then, since f is 1-Lipschitz,

$$|f(y_1) - f(y_2)| \geq 8M \operatorname{rad} B/10$$

for each $y_i \in B_i$ and each $i = 1, 2$. In particular, there exists $i \in 1, 2$ such that

$$|f(y_i) - f_{2B}| \geq 4M \operatorname{rad} B/10$$

for each $y_i \in B_i$. Therefore,

$$\fint_{2B} |f - f_{2B}| \, \mathrm{d}\mu \geq \frac{1}{\mu(2B)} \int_{B_i} |f - f_{2B}| \, \mathrm{d}\mu \geq \frac{2M \operatorname{rad} B \mu(B_i)}{5\mu(2B)} \tag{4.1}$$

Now, $\operatorname{rad} B_i = M \operatorname{rad} B/10$ and so, since μ is doubling,

$$C(\mu)^n \mu(B_i) \geq \mu(2B)$$

for $n = -\log_2 M/10 + 2$. That is, if $\delta = \log_2 C > 0$ depending on $C(\mu)$ such that

$$\frac{\mu(B_i)}{\mu(2B)} \geq CM^\delta$$

Combining this with eq. (4.1) completes the first part of the proof.

The second part of the lemma simply follows from the Lebesgue differentiation theorem. □

Proposition 4.3. *Let (X, d, μ) be a PI space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz satisfying eq. (2.1) for all $x \in U \subset X$. Then any $\tilde{A}(\phi)$ subset of U has μ measure zero.*

Proof. Since the definition of an $\tilde{A}(\phi)$ set is invariant under scaling ϕ, we may suppose that ϕ is 1-Lipschitz.

For each $x \in U$, the function

$$D \in \mathbb{S}^{n-1} \mapsto \operatorname{Lip}(D \cdot \phi, x)$$

is Lipschitz and positive by assumption. Therefore, it is bounded below by some $\lambda_x > 0$ for each $x \in U$. The map $x \mapsto \lambda_x$ is Borel and so we may decompose U into sets $U_i = \{x : \lambda_x > 1/i\}$, $i \in \mathbb{N}$. Since it suffices to prove that any $\tilde{A}(\phi)$ subset of each U_i has μ measure zero, we may suppose that $U = U_i$ for some $i \in \mathbb{N}$ and set $\lambda = 1/i$.

For $0 < \theta < 1$ and $w \in \mathbb{S}^{n-1}$, fix a $C(w, 90 - \theta)$-null set $S \subset U$ and set $f = w \cdot \phi\colon X \to \mathbb{R}$. We claim that

$$\rho = 1_{S^c} + \theta 1_S$$

is an upper gradient of f. Indeed, if $\gamma\colon [0, l] \to X$ is parametrised by arc-length, then by Lemma 3.6, $(\phi \circ \gamma)' \notin \operatorname{int}(C(w, \theta))$ for almost every $t \in \gamma^{-1}(S)$. That is,

$$|(f \circ \gamma)'(t)| = |w \cdot (\phi \circ \gamma)'(t)| \leq \theta \|(\phi \circ \gamma)'(t)\| \leq \theta \operatorname{Lip} \phi = \theta$$

for almost every $t \in \gamma^{-1}(S)$. Therefore, by the fundamental theorem of calculus,

$$|f(\gamma(l)) - f(\gamma(0))| \leq \int_0^l |(f \circ \gamma)'(t)| \leq \int_{\gamma^{-1}(S)} \theta + \int_{\gamma^{-1}(X \setminus S)} 1 = \int_0^l \rho(t) \, \mathrm{d}t,$$

as required.

By applying Lemma 4.1, we find a $C, \eta \geq 1$ depending only on PI(μ) such that, for μ almost every $x \in S$,

$$0 < \lambda \leq \operatorname{Lip}(f, x) \leq C\rho(x)^{1/\eta} = C\theta^{1/\eta}.$$

This is impossible if θ is sufficiently small, and so we must have $\mu(S) = 0$. Precisely, any $C(w, \lambda^\eta/C)$-null subset of U has μ measure zero. Therefore, any $\tilde{A}(\phi)$ subset of U has μ measure zero, as required. □

We complete this section by summarising the main result.

Theorem 4.4. *Let (X, d, μ) be a PI space. Suppose that $U \subset X$ and $\phi \colon X \to \mathbb{R}^n$ is Lipschitz such that eq. (2.1) holds for all $x \in U$. Then there exists a countable Borel decomposition $U = \cup_i A_i$ such that each $\mu|_{A_i}$ has n independent Alberti representations.*

In particular, if there exists an $N \in \mathbb{N}$ such that, for any $U \subset X$ with $\mu(U) > 0$, $\mu|_U$ can have at most N independent Alberti representations, then X is a Lipschitz differentiability space.

Proof. The first part follows by combining Theorem 3.9 and Proposition 4.3.

By combining the hypotheses of the second part with the first part of the theorem, we precisely statisfy the hypotheses of Proposition 2.7. The conclusion follows. □

5. Gromov–Hausdorff convergence

There is a standard notion of *convergence of metric spaces* introduced by Gromov.

A *pointed metric space* (X, x) is simply a metric space X and a point $x \in X$.

Definition 5.1. A sequence of pointed metric spaces (X_n, x_n) *Gromov–Hausdorff converge* to a metric space (X, x) if there is a sequence of maps $\iota_i \colon X \to X_i$ with $\iota_i(x) = x_i$ such that, for all $R > 0$,

$$\sup\{d_{X_i}(\iota_i(y), \iota_i(z)) - d_X(y, z) : y, z \in B(x, R)\} \to 0,$$

(that is, the ι_i uniformly approximate an isometry on balls) and, for all $\delta > 0$,

$$\sup\{d(y, \iota_i(B(x, R + \delta))) : y \in B(x_i, R)\} \to 0$$

(that is, the ι_i uniformly approximate a surjection ob balls).

Theorem 5.2 (Gromov)**.** *Given any sequence (X_i, x_i) of pointed N_D doubling metric spaces, there exists a subsequence that Gromov–Hausdorff converges to some N_D doubling pointed metric space (X, x).*

The proof is an Arzelà-Ascoli type of argument. For each $\epsilon > 0$ we can approximate $B(x_i, R)$ by a maximal ϵ-separated net. Because the metric spaces are uniformly doubling, the size of this net only depends on ϵ, and not i. By taking a subsequence, we can find a convergent subsequence of these nets. By taking a diagonal subsequence over $\epsilon \to 0$ and then $R \to \infty$ gives the required result. See [**MR1835418**].

Definition 5.3. Let (X, d) be a doubling metric space and $x \in X$. A *Gromov–Hausdorff* tangent of X at x is any pointed metric space $(X_\infty, d_\infty, x_\infty)$ for which there exists $r_i \to 0$ such that $(X, d/r_i, x) \to (X_\infty, d_\infty, x_\infty)$. The set of all such tangents is denoted $\operatorname{Tan}(X, x)$.

Note that, for any $r_i \to 0$, by the Gromov compactness theorem, there exists a subsequence r_{i_k} for which the rescalings converge.

Let $(X_\infty, x_\infty) \in \operatorname{Tan}(X, x)$ and $r_i \to 0$ such that $(X, d/r_i, x) \to (X_\infty, d_\infty, x_\infty)$ via the maps ι_i. For any Lipschitz $\phi\colon X \to \mathbb{R}^n$, the functions

$$\phi_i\colon (X, d/r_i) \to \mathbb{R}^n$$

$$\phi_i(y) = \frac{\phi(y) - \phi(x)}{r_i}$$

are $\operatorname{Lip}\phi$-Lipschitz. Consequently, by another Arzelà-Ascoli type argument, after passing to a further subsequence, there exists a Lipschitz $\phi_\infty\colon X_\infty \to \mathbb{R}^n$ such that, for any $R > 0$,

$$\phi_i(\iota_i(y)) \to \phi_\infty(y)$$

uniformly on $B(x, R)$. We will denote by $\operatorname{Tan}(X, d, x, \phi)$ the collection of all such $(X_\infty, d_\infty, x_\infty, \phi_\infty)$.

These are related to Alberti representations in the obvious way.

Lemma 5.4. *Let (X, d) be a metric space, $\gamma \in \Gamma(X)$ and $\phi\colon X \to \mathbb{R}^n$ Lipschitz with $\gamma(t) = x_0$. Suppose that $(\phi \circ \gamma)'(t_0)$ exists and t_0 is a density point of* $\operatorname{dom}\gamma$. *Then for any*

$$(X_\infty, d_\infty, x_\infty, \phi_\infty) \in \operatorname{Tan}(X, d, x_0, \phi),$$

there exists a bi-Lipschitz line $l\colon \mathbb{R} \to X_\infty$ *such that*

$$\phi_\infty(l(t)) = \phi_\infty(x_\infty) + t(\phi \circ \gamma)'(t_0).$$

We say that this line passes through x_∞ and is parallel to $(\phi \circ \gamma)'(t_0)$.

In particular, if μ has n independent Alberti representations, then for μ almost every $x \in X$, there exists linearly independent $v_1, \ldots, v_n\mathbb{R}^n$ such that, for any

$$(X_\infty, d_\infty, x_\infty, \phi_\infty) \in \operatorname{Tan}(X, d, x_0, \phi),$$

X_∞ contains lines $l_1, \ldots, l_n$ passing through x_∞ that are tangent to $v_1, \ldots, v_n$ respectively.

Proof. This is a proof by picture. □

We also require the following.

Proposition 5.5. *Let (X, d, μ) be a doubling metric measure space. For μ almost every $x_0 \in X$, if*

$$(X_\infty, d_\infty, x_\infty, \phi_\infty) \in \operatorname{Tan}(X, d, x_0, \phi),$$

and $x \in X_\infty$, then

$$(X_\infty, d_\infty, x, \phi_\infty - \phi_\infty(x)) \in \operatorname{Tan}(X, d, x_0, \phi),$$

By combining the previous two results we have the following.

Corollary 5.6. *Let (X, d, μ) be a doubling metric measure space with n independent Alberti representations via a Lipschitz map $\phi\colon X \to \mathbb{R}^n$. Then for μ almost every $x \in X$, there exists linearly independent $v_1, \ldots, v_n \in \mathbb{R}^n$ such that, for any*

$$(X_\infty, d_\infty, x_\infty, \phi_\infty) \in \operatorname{Tan}(X, d, x_0, \phi),$$

and any $x \in X_\infty$, X_∞ contains lines $l_1, \ldots, l_n$ passing through x that are tangent to $v_1, \ldots, v_n$ respectively. In particular, ϕ_∞ maps X_∞ onto $\mathbb{R}^n$.

To conclude we have the following theorem.

Theorem 5.7. *Let (X, d, μ) be a doubling metric measure space. There exists a $N \in \mathbb{N}$ depending on the doubling constant such that the following is true. If there exists a $U \subset X$ of positive measure such that $\mu|_U$ has n independent Alberti representations, then $n \leq N$.*

Proof. Suppose that the Alberti representations of $\mu|_U$ are independent with respect to $\phi\colon X \to \mathbb{R}^n$. By Corollary 5.6, for μ almost every $x \in U$ and any

$$(X_\infty, d_\infty, x_\infty, \phi_\infty) \in \mathrm{Tan}(X, d, x_0, \phi),$$

ϕ_∞ maps X_∞ onto $\mathbb{R}^n$. Since U has positive measure, at least one such X_∞ and ϕ_∞ exists.

We know that (X_∞, d_∞) is N_D-doubling. In particular, for any $R > 0$ and $n \in \mathbb{N}$, $B(x_\infty, R)$ is contained in N_D^n balls of radius $2^{-n}R$. In particular, its Hausdorff dimension is bounded above by CN_D. Since f is Lipschitz and $f(X_\infty) = \mathbb{R}^n$, we have

$$n = \dim \mathbb{R}^n \leq \dim X_\infty \leq CN_D,$$

as required. □

Combining Theorem 4.4 and Theorem 5.7 completes the proof of Cheeger's theorem.

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