EXERCISE SHEET 2

Exercise 1. *Let (X, d) be a metric space and $f\colon X \to \mathbb{R}$ (note that we do not assume any regularity on f).*

i) *For $\epsilon > 0$ let C_ϵ be the set of all $x \in X$ for which there exists a $\delta > 0$ such that, for all $y, z \in X$ with $d(x, y), d(x, z) < \delta$, $|f(y) - f(z)| < \epsilon$. Show that C_ϵ is open.*
ii) *Show that the set of points where f is continuous is a countable intersection of open sets.*
iii) *Suppose that $X = \mathbb{R}$. Show that the set of points where f is differentiable is a Borel set.*

Exercise 2. *A subset S of a metric space X is* porous *if for every $x \in S$ there exists a $\lambda > 0$ and a sequence $x_n \to x$ such that*

$$B(x_n, \lambda d(x, x_n)) \cap S = \emptyset$$

for every $n \in \mathbb{N}$. (Recall that this notion was used in the proof of Rademacher's theorem.)

Prove that any porous subset of a doubling metric measure space has measure zero.

Recall that for a metric space X, $S \subset X$ and $\alpha \geq 0$, the *α-dimensional outer Hausdorff measure of S* is defined by

$$\mathcal{H}^\alpha(S) = \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} (\operatorname{diam} S_i)^\alpha : S \subset \bigcup_{i \in \mathbb{N}} S_i,\ \operatorname{diam} S_i < \delta \right\}.$$

For any metric space X and any $\alpha \geq 0$, $\mathcal{H}^\alpha$ is a Borel measure.

Exercise 3. *Let X, Y be metric spaces, $f\colon X \to Y$ and $\alpha, L \geq 0$.*

i) *Suppose that f is L-Lipschitz and $S \subset X$. Prove that $\mathcal{H}^\alpha(f(S)) \leq L^\alpha \mathcal{H}^\alpha(S)$.*
ii) *For $R > 0$ suppose that, for each $x \in S$ and $y \in X$ with $d(x, y) < R$, $d(f(x), f(y)) < Ld(x, y)$. Prove that $\mathcal{H}^\alpha(f(S)) \leq L^\alpha \mathcal{H}^\alpha(S)$.*
iii) *Define*

$$\operatorname{Lip}(f, x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)},$$

the pointwise Lipschitz constant *of f at x. Suppose that $\operatorname{Lip}(f, x) < L$ for each $x \in X$. Prove that $\mathcal{H}^\alpha(f(X)) \leq L^\alpha \mathcal{H}^\alpha(X)$. (You may assume that $x \mapsto \operatorname{Lip}(f, x)$ is Borel, along with other similar statements.)*

Exercise 4. *Let $\gamma\colon [a, b] \to X$ be Lipschitz.*

i) *Prove that $\operatorname{len}(\gamma) \geq d(\gamma(b), \gamma(a))$.*
ii) *Prove that the arc-length parametrisation γ^* of any Lipschitz $\gamma\colon [a, b] \to X$ exists and is 1-Lipschitz.*
iii) *Prove that, for any $s \leq t \in [0, \operatorname{len}(\gamma)]$, $\operatorname{len}(\gamma^*|_{[s,t]}) = t - s$.*
iv) *From now on, suppose that γ is injective. Prove that $\operatorname{len}(\gamma^*|_{[s,t]}) = \mathcal{H}^1(\gamma^*([s, t]))$.*
v) *Deduce that, for any Borel $S \subset [0, \operatorname{len}(\gamma)]$, $\mathcal{H}^1(\gamma^*(S)) = \mathcal{L}^1(S)$.*
vi) *Show that this may be false if γ is not injective.*

Exercise 5. *Let X, Y be metric spaces, $S \subset X$ and let $f\colon S \to Y$ be L-Lipschitz. Suppose that Y is complete. Prove that there exists a unique L-Lipschitz extension of f defined on the closure of S. Prove that this is not necessarily the case if Y is not complete.*

Exercise 6. *Let X, Y be metric spaces and for each $n \in \mathbb{N}$, $f_n\colon X \to Y$ L-Lipschitz. Suppose that $f_n \to f$ pointwise. Prove that f is L-Lipschitz.*

Exercise 7. *The space ℓ_∞ consists of all bounded real sequences equipped with the supremum norm. It is a Banach space (that is, a complete vector space; if you have never proved that it is complete, you should do so now).*

Let (X, d) be a separable metric space, $x_0, x_1, x_2, \ldots$ a dense sequence and define $\iota\colon X \to \ell_\infty$ by

$$\iota(x) = (d(x, x_1) - d(x_1, x_0), d(x, x_2) - d(x_2, x_0), d(x, x_3) - d(x_3, x_0), \ldots).$$

Prove that ι is well defined and an isometry.

This shows that any separable metric space can be isometrically embedded into a Banach space. However, this Banach space is huge (it is not separable). We can do a little better. Let $\tilde{X}$ be the closed linear span of $\iota(X)$. That is, $\tilde{X}$ is the closure of the set

$$\left\{ \sum_{i=1}^{n} \lambda_i y_i : n \in \mathbb{N},\ \lambda_i \in \mathbb{R},\ y_i \in \iota(X) \right\}.$$

Certainly $\tilde{X}$ is a closed subspace (and so a Banach space). Prove that it is separable.