METRIC GEOMETRY, FALL 2018
EXERCISE 4, IDEAS FOR SOLUTIONS.

Exercise 1. Suppose that $f\colon Y \to \mathbb{R}$ is 1-Lipschitz, and let L be the bilipschitz constant of ι. We consider the function $g\colon X \to \mathbb{R}$ defined by $g = (f \circ \iota)/L$. Then g is a 1-Lipschitz function.

Let ρ be an upper gradient of f, let $x, x' \in X$, and suppose $\gamma\colon [a,b] \to X$ is a rectifiable curve joining x and x'. Then $\iota \circ \gamma$ is a rectifiable curve joining $\iota(x)$ and $\iota(x')$. We now have

$$\begin{aligned} \left|f(\iota(x)) - f(\iota(x'))\right| &\leq \int_{\iota\circ\gamma} \rho \,\mathrm{d}s \\ &= \int_0^{\mathrm{len}(\iota\circ\gamma)} \rho \circ (\iota\circ\gamma)^*(t)\,\mathrm{d}t. \end{aligned}$$

Recalling the function $l_\gamma(t) = \mathrm{len}(\gamma|_{[a,t]})$, a straightforward bilipschitz estimate on the definition on len yields

$$L^{-1}\,\mathrm{len}(\gamma|_{[s,t]}) \leq \mathrm{len}(\iota\circ\gamma|_{[s,t]}) \leq L\,\mathrm{len}(\gamma|_{[s,t]})$$

for all $s, t \in [a,b]$, and therefore in particular,

$$L^{-1} l_\gamma(t) \leq l_{\iota\circ\gamma}(t) \leq L l_\gamma(t)$$

for all $t \in [a,b]$. Hence, it is relatively easy (see the methods used in Ex. 4 of the second exercise sheet) to find a bijective increasing $\sigma\colon [0, \mathrm{len}(\gamma)] \to [0, \mathrm{len}(\iota\circ\gamma)]$ for which $l_{\iota\circ\gamma} = \sigma \circ l_\gamma$. Moreover, by the identity $l_\gamma(s) - l_\gamma(t) = \mathrm{len}(\gamma|_{[s,t]})$ for $s > t$, it follows easily that σ is L-bilipschitz.

Now, we have

$$(\iota\circ\gamma)^* \circ \sigma \circ l_\gamma = (\iota\circ\gamma)^* \circ l_{\sigma\circ\gamma} = \iota\circ\gamma = \iota\circ\gamma^*\circ l_\gamma.$$

Therefore, by uniqueness of the arc-length parametrization, we see that

$$(\iota\circ\gamma)^* = \iota\circ\gamma^*\circ\sigma^{-1}.$$

With this, we may finally continue our previous estimate on the expression $|f(\iota(x)) - f(\iota(x'))|$ by a bilipschitz change of variables

$$\begin{aligned} \left|f(\iota(x)) - f(\iota(x'))\right| &\leq \int_0^{\mathrm{len}(\iota\circ\gamma)} \rho\circ(\iota\circ\gamma)^*(t)\,\mathrm{d}t \\ &= \int_0^{\mathrm{len}(\gamma)} \rho\circ\iota\circ\gamma^*(t')\sigma'(t')\,\mathrm{d}t' \\ &\leq L\int_0^{\mathrm{len}(\gamma)} \rho\circ\iota\circ\gamma^*(t')\,\mathrm{d}t' \\ &= \int_\gamma L\cdot\rho\circ\iota\,\mathrm{d}s. \end{aligned}$$

Dividing both sides by L, we conclude that $\rho\circ\iota$ is an upper gradient of g.

Now, our assumption on X yields that

$$\operatorname{Lip}(g,x)^\eta \le C\rho \circ \iota(x)$$

for μ-almost every $x \in X$. Since $g = (f \circ \iota)/L$, We may compute by the definition of Lip that

$$\begin{aligned}\operatorname{Lip}(g,x) &= \limsup_{x'\to x} \frac{\left|\frac{f(\iota(x))}{L} - \frac{f(\iota(x'))}{L}\right|}{d(x,x')}\\ &= \frac{1}{L}\limsup_{x\to x'} \frac{|f(\iota(x)) - f(\iota(x'))|}{d(x,x')}\\ &\ge \frac{1}{L^2}\limsup_{x\to x'} \frac{|f(\iota(x)) - f(\iota(x'))|}{d(\iota(x),\iota(x'))}\\ &= \frac{1}{L^2}\operatorname{Lip}(f,\iota(x)),\end{aligned}$$

where the last step uses the fact that ι is a homeomorphism. Hence, if $y = \iota(x)$ and x satisfies the previous $\operatorname{Lip}(g,x)$-estimate, we obtain

$$\operatorname{Lip}(f,y)^\eta \le CL^{2\eta}\rho(y).$$

This holds with ν-almost every y, since ν is the pushforward of μ by a bijective map ι and the $\operatorname{Lip}(g,x)$-estimate holds for μ-almost every $x \in X$. Hence, we may select $\eta' = \eta$ and $C' = L^{2\eta}C$, and the claim holds.

Exercise 2. We suppose that (X,d) is a Lipschitz differentiability space and that (U,φ) is a chart in X, with $\varphi\colon X \to \mathbb{R}^m$. Let $\psi\colon X \to \mathbb{R}^n$ be a Lipschitz function such that $\operatorname{Lip}(v\cdot\psi,x) > 0$ for every $x \in U$ and $v \in \mathbb{R}^n \setminus \{0\}$.

Since (U,φ) is a chart in a Lipschitz differentiability space, we find a point $x_0 \in U$ and a unique linear differential map $D\psi(x_0) : \mathbb{R}^m \to \mathbb{R}^n$ with respect to φ (see Ex. 5 of the third exercise sheet). Suppose that $n > m$. Then $D\psi(x_0)$ is not surjective, and we therefore find a $v \in \mathbb{R}^n \setminus \{0\}$ which is orthogonal to the entire image of $D\psi(x_0)$.

By our assumption, we have

$$0 < \operatorname{Lip}(v\cdot\psi, x_0) = \limsup_{x\to x_0} \frac{|v\cdot(\psi(x) - \psi(x_0))|}{d(x,x_0)}.$$

However, we may also compute

$$\begin{aligned}&\limsup_{x\to x_0} \frac{|v\cdot(\psi(x) - \psi(x_0))|}{d(x,x_0)}\\ &\quad= \limsup_{x\to x_0} \frac{|v\cdot(\psi(x) - \psi(x_0) - D\psi(x_0)(\varphi(x)-\varphi(x_0)))|}{d(x,x_0)}\\ &\quad\le |v|\limsup_{x\to x_0} \frac{|\psi(x) - \psi(x_0) - D\psi(x_0)(\varphi(x)-\varphi(x_0))|}{d(x,x_0)}\\ &\quad= 0.\end{aligned}$$

This is a contradiction. Hence, the assumption that $\operatorname{Lip}(v\cdot\psi,x) > 0$ for every $x \in U$ and $v \in \mathbb{R}^n \setminus \{0\}$ necessarily requires that $n \le m$, where m is the dimension of the target of the chart (U,φ).

Exercise 3. Recalling the definitions given, we have that $\mathcal{C}$ is the set of non-empty closed bounded subsets of (X, d), and

$$d_H(C, D) = \inf\{r \geq 0 : C \subset B(D, r) \text{ and } D \subset B(C, r)\}$$

for $C, D \in \mathcal{C}$, where $B(C, r)$ is the closed ball of radius r around the set C.

(i): Suppose that $C, D \in \mathcal{C}$. It is clear that $d_H(C, D) = d_H(D, C)$. It is also clear that $B(C, 0) = C$, and hence $d_H(C, C) = 0$.

On the other hand, suppose that $C \neq D$. Then one of the two sets has a point x not in the other – by symmetry we may assume $x \in C$ and $x \neq D$. Since D is closed and bounded, we have $0 < d(x, D) < \infty$. Hence, for any $r < d(x, D)$, we have $x \notin B(D, r)$, and we therefore obtain $d_H(C, D) \geq d(x, D) > 0$. In conclusion, $d_H(C, D) = 0$ if and only if $C = D$.

Remains to verify the triangle inequality. For this, let $C, D, E \in \mathcal{C}$. Let $\varepsilon > 0$, $r = d_H(C, D) + \varepsilon$ and $r' = d_H(D, E) + \varepsilon$. Then $C \subset B(D, r)$ and $D \subset B(E, r')$. It follows that $C \subset B(E, r + r')$. We deduce similarly that $E \subset B(C, r + r')$. Hence, $d_H(C, E) \leq r + r' = d_H(C, D) + d_H(D, E) + 2\varepsilon$. By letting $\varepsilon \to 0$, we obtain the triangle inequality.

(ii): Let for example $X = \mathbb{R}$, $C = B(0, 1)$ and $D = B(0, 1) \cap \mathbb{Q}$. Then $D \subset C = B(C, 0)$, and for every $r > 0$, we have $C \subset B(D, r)$. It follows that $d_H(C, D) = 0$, but $C \neq D$.

Remark: In a similar manner, d_H is not a metric on non-empty closed subsets, since two sets can have distance ∞ in this case.

Exercise 4. As described in the exercise, suppose that X is complete and that (C_i) is a Cauchy sequence of $\mathcal{C}$ in d_H. We may suppose by moving to a subsequence that $d_H(C_m, C_n) < 2^{-n}$ for all $m \geq n$. We let C be the set of points $c \in X$ for which there exist points $c_n \in C_n$ so that the sequence c_n converges to c.

(i): Pick $c_1 \in C_1$ completely arbitrarily. Now by our assumption we have $C_1 \subset B(C_2, 2^{-1})$, and therefore there exists $c_2 \in C_2$ for which $d(c_1, c_2) \leq 2^{-1}$. Similarly, we find $c_3 \in C_3$ for which $d(c_2, c_3) \leq 2^{-2}$, and so on. Since $\sum_{i=1}^{\infty} 2^{-i} = 1 < \infty$, it is easily seen that (c_n) is a Cauchy sequence. Since X is complete, (c_n) has a limit c, which therefore is in C. Hence, C is non-empty.

Now suppose $c \in C$ is given by a sequence c_n. Then there exists some n for which $d(c, c_n) \leq 1$. It follows that $c \in B(C_n, 1)$ for some n. We conclude that

$$C \subset \bigcup_{n=1}^{\infty} B(C_n, 1).$$

However, since we could assume that $d_H(C_n, C_1) < 1/2$ for all n, we obtain that $B(C_n, 1) \subset B(C_1, 3/2)$ for all n, and therefore

$$C \subset B(C_1, 3/2).$$

Hence, C is bounded.

Remains to check that C is closed. For this, let (c^m) be a sequence in C converging to $c \in X$. We wish to show $c \in C$. For every c^m, let $c_n^m \in C_n$ converge to c^m as $n \to \infty$. Then for every m, we find a $N(m)$ so that $d(c_n^m, c^m) < 1/m$ when $n \geq N(m)$. We may assume $N(m + 1) > N(m)$ for all m. Now, we select $c_n \in C_n$ so that $c_n = c_n^{m_n}$, where $m_n \geq 0$ is the integer satisfying $N(m_n) \leq n < N(m_n+1)$. It then follows that

$$d(c_n, c) \leq d(c_n^{m_n}, c^{m_n}) + d(c^{m_n}, c) < \frac{1}{m_n} + d(c^{m_n}, c).$$

Moreover, $m_n \to \infty$ as $n \to \infty$, as $m_n \geq m$ whenever $n \geq N(m)$. Hence, $d(c_n, c) \to 0$ as $n \to \infty$. Now by definition of C we get $c \in C$, and we therefore conclude that C is closed.

(ii): Let $\varepsilon > 0$. We select N_1 to be large enough that $2^{-N_1+1} < \varepsilon$. Let $c \in C$ and $n \geq N_1$. We may by definition of C pick a Cauchy sequence (c_m) converging to c, with $c_m \in C_m$ for every m. Then there exists $n' \geq n$ so that $d(c, c_{n'}) < \varepsilon - 2^{-N_1+1}$. Since $d_H(C_{n'}, C_{n'-1}) < 2^{n'-1}$, we find $z_{n'-1} \in C_{n'-1}$ with $d(c_{n'}, z_{n'-1}) < 2^{n'-1}$. We then find $z_{n'-2} \in C_{n'-2}$ so that $d(z_{n'-1}, z_{n'-2}) < 2^{n'-2}$, and so on. Eventually we end up with $z_n \in C_n$, and now

$$\begin{aligned} d(c, z_n) &\leq d(c, c_{n'}) + d(c_{n'}, z_{n'-1}) + d(z_{n'-1}, z_{n'-2}) + \ldots + d(z_{n+1}, z_n) \\ &< \left(\varepsilon - 2^{-N_1+1}\right) + \sum_{i=n}^{n'-1} 2^{-i} \\ &< \left(\varepsilon - 2^{-N_1+1}\right) + \sum_{i=N_1}^{\infty} 2^{-i} \\ &< \left(\varepsilon - 2^{-N_1+1}\right) + 2^{-N_1+1} \\ &< \varepsilon. \end{aligned}$$

Hence, $c \in B(\varepsilon, C_n)$.

(iii): Suppose then that $N > 0$ and $c^* \in C_N$. We construct a sequence of $c_n \in C_n$ as follows. For $n < N$, we let c_n be an arbitrary point of C_n. For $n = N$, we select $c_n = c^*$. If $n \geq N$, we note that since $d_H(C_{n+1}, C_n) < 2^{-n}$, there exists $c_{n+1} \in C_{n+1}$ for which $d(c_{n+1}, c_n) < 2^{-n}$. This lets us inductively select c_n for $n > N$.

Now the sequence (c_n) is Cauchy, since $d(c_n, c_m) < 2^{-n+1}$ for $m \geq n$. By completeness of X, it has a limit $c \in X$, and by definition of C we have $c \in C$. Moreover, $d(c^*, c) = d(c_N, c) \leq 2^{-N+1}$. Hence, we obtain that $C_N \subset B(C, 2^{-N+1})$, and therefore that $d_H(C_n, C) \to 0$. In conclusion, $(\mathcal{C}, d_H)$ is complete.

Exercise 5. Suppose that X is totally bounded. As described in the hint, let $\varepsilon > 0$, use total boundedness of X to select $x_1, \ldots, x_m$ for which $X = B(x_1, \varepsilon) \cup \ldots \cup B(x_m, \varepsilon)$, and then let $\mathcal{B}$ be the collection

$$\mathcal{B} = \left\{ \bigcup_{j=1}^{k} B(x_{j_i}, \varepsilon) : 1 \leq j_i \leq m, 1 \leq k \leq m \right\}.$$

Suppose then that C is a nonempty closed bounded subset of X. We let C_ε be the union of those balls $B(x_i, \varepsilon)$ which C intersects. Then $C_\varepsilon \in \mathcal{B}$. We clearly have $C \subset C_\varepsilon$. Moreover, if C intersects $B(x_i, \varepsilon)$, then there exists $c \in C \cap B(x_i, \varepsilon)$, and we have $B(x_i, \varepsilon) \subset B(c, 2\varepsilon)$. Hence, $C_\varepsilon \subset B(C, 2\varepsilon)$, and therefore $d_H(C, C_\varepsilon) < 2\varepsilon$. We conclude that every $C \in \mathbb{C}$ is at most

d_H-distance 2ε away from a set of $\mathcal{B}$, and therefore that $(\mathcal{C}, d_H)$ is totally bounded.

Exercise 6. For the first direction, let X be compact, and let $C_n \in \mathcal{C}$.

(i): Consider the function $f_n : x \mapsto d(x, C_n)$ on X. Since X is compact, there exists a maximal distance $d < \infty$ between any two points of X, and therefore the image of f_n is contained in $[0, d]$ for all n.

We wish to show that f_n is 1-Lipschitz. Let $x, y \in X$, and suppose $f_n(x) \geq f_n(y)$. Since C_n is closed nonempty and X is compact, there exists $y' \in C_n$ so that $d(y, C_n) = d(y, y')$. Now

$$\begin{aligned} |f_n(x) - f_n(y)| &= f_n(x) - f_n(y) \\ &= d(x, C_n) - d(y, C_n) \\ &\leq d(x, y') - d(y, y') \\ &\leq d(x, y). \end{aligned}$$

Hence, f_n is 1-Lipschitz. Now the Arzela–Ascoli -theorem lets us obtain a 1-Lipschitz limit f of f_n after moving to a subsequence.

(ii): We let $C = f^{-1}\{0\}$. Since f is continuous, C is closed, and since X is compact, C is bounded. Moreover, there exists $c_n \in C_n$ for every n, and therefore $f_n(c_n) = 0$ for every n. By compactness of X, we may select a subsequence of c_n which converges to some $c \in X$. Since all f_n are 1-lipschitz, $|f_n(c)| \leq d(c, c_n)$. It follows that $f(c) = 0$, and therefore C is nonempty. Hence, $C \in \mathcal{C}$.

Next, we show that in fact, $f(x) = d(x, C)$ for all $x \in X$. Suppose that $x \in X$. Since C is closed nonempty and X is compact, there is a point $c \in C$ closest to x. Since f is a pointwise limit of 1-Lipschitz maps, f is 1-Lipschitz. Therefore, using the fact that $f(c) = 0$ and $f \geq 0$, we obtain

$$f(x) = |f(x) - f(c)| \leq d(x, c) = d(x, C).$$

For the converse inequality, for every $n \geq 0$, let $c_n \in C_n$ be the closest point to x in C_n as above. Then similarly as before, there is a subsequence (c_{n_i}) which converges to some $c' \in X$, and $c' \in C$ since the fact that f_n are 1-Lipschitz implies $|f_{n_i}(c)| \leq d(c, c_{n_i})$. Now

$$\begin{aligned} d(x, C) &\leq d(x, c') \\ &= \lim_{i \to \infty} d(x, c_{n_i}) \\ &= \lim_{i \to \infty} f_{n_i}(x) \\ &= f(x). \end{aligned}$$

We conclude that $f(x) = d(x, C)$ for all $x \in X$.

Now, we note that since f_n are continuous and X is compact, we in fact have $f_n \to f$ uniformly. Hence, for every $\varepsilon > 0$, there exists N_ε for which $|f - f_n| < \varepsilon$ everywhere on X. Suppose that $n \geq N_\varepsilon$. Since $f|_C = 0$, we have $f_n|_C < \varepsilon$. Now it follows that $d(x, C_n) = f_n(x) < \varepsilon$ for every $x \in C$, and therefore $C \subset B(C_n, \varepsilon)$. Similarly, since $f_n|_{C_n} = 0$, we have $f|_{C_n} < \varepsilon$. Since we showed that $f(x) = d(x, C)$ for all $x \in X$, we obtain that $d(x, C) = f(x) < \varepsilon$ for every $x \in C_n$, and therefore $C_n \subset B(C, \varepsilon)$. We conclude that $d_H(C_n, C) \leq \varepsilon$ for all $n \geq N_\varepsilon$, and therefore $C_n \to C$ in d_H.

Now for the other part of the exercise, suppose that X, Y are compact metric spaces and $f_n\colon X \to Y$ are L-lipschitz. Let C_n be the graph of f_n, comprised of points of the form $(x, f(x))$ in $X \times Y$.

(iii): The maps $g_n\colon X \to X \times Y$ defined by $g_n = (\mathrm{id}_X, f_n)$ are continuous, since their component functions are continuous. Hence, for every n, we have that $C_n = g_n(X)$ is a continuous image of a compact space, and therefore C_n is compact.

(iv): By the previous two exercises, $\mathcal{C}(X \times Y)$ is complete and totally bounded under d_H, and therefore compact. Therefore, by moving to a subsequence, we may assume $C_n \to C \in \mathcal{C}(X \times Y)$ in d_H. We wish to show that C is the graph of some function $f\colon X \to Y$.

There are two ways in which C can fail to be a graph of a function: either if some point $x \in X$ has multiple images (points of the form (x, y) where $y \in Y$) in C, or if some $x \in X$ has no images in C. Consider first the case of multiple images: suppose that (x, y) and (x, y') are both in C. Since C_n converge to C in d_H, we find $(x_n, f_n(x_n)) \in C_n$ and $(x'_n, f_n(x'_n)) \in C_n$ which converge to (x, y) and (x, y'), respectively. Now

$$\begin{aligned} d(y, y') &\leq d((x, y), (x, y')) \\ &= \lim_{n\to\infty} d((x_n, f_n(x_n)), (x'_n, f_n(x'_n))) \\ &\leq \limsup_{n\to\infty} d(x_n, x'_n) + d(f_n(x_n), f_n(x'_n)) \\ &\leq \limsup_{n\to\infty} (L+1) d(x_n, x'_n) \\ &= 0. \end{aligned}$$

Hence, $y = y'$. Therefore, there is no point $x \in X$ which has multiple images in C.

Consider now the other case: suppose to the contrary that $x \in X$ has no images in C, that is, $C \cap (\{x\} \times Y) = \emptyset$. Since C and $\{x\} \times Y$ are compact, we can therefore select an $\varepsilon > 0$ for which $B(C, \varepsilon)$ and $B(\{x\} \times Y, \varepsilon)$ are disjoint. However, since $C_n \cap (\{x\} \times Y)$ is nonempty for every n, we now have that no C_n is contained in $B(C, \varepsilon)$. This contradicts d_H-convergence of C_n to C. Therefore, $C \cap (\{x\} \times Y) \neq \emptyset$ for every $x \in X$.

We have obtained that for every x, there is a unique $(x, y) \in C$. Therefore, C is the graph of a function, which we may denote by f.

(v): We now wish to show that $f_n \to f$ uniformly. Select N_ε such that $d_H(C_n, C) < \varepsilon$ whenever $n \geq N_\varepsilon$. Hence, for every $x \in X$ and $n \geq N_\varepsilon$, there exists $x' \in X$ for which $d((x, f(x)), (x', f_n(x'))) < \varepsilon$. It follows that $d(x, x') < \varepsilon$ and $d(f(x), f_n(x')) < \varepsilon$. We may therefore estimate

$$\begin{aligned} d(f(x), f_n(x)) &\leq d(f(x), f_n(x')) + d(f_n(x'), f_n(x)) \\ &\leq d(f(x), f_n(x')) + L d(x', x) \\ &< (1 + L)\varepsilon. \end{aligned}$$

This shows uniform convergence of f_n to f (which also consequently implies that f is L-Lipschitz).