RADEMACHER'S THEOREM FOR METRIC MEASURE SPACES

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1. Introduction

Rademacher's theorem states that any Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable Lebesgue almost everywhere. This is a generalisation of a result of Lebesgue who proved it for $n = 1$. It is a fundamental result in geometric measure theory. For example, it shows that any rectifiable set possesses a weak tangent plane at almost every point. Moreover the statement itself is very interesting in its own right; the fact that a seemingly rather simple condition can impose such strong regularity is quite remarkable. Rademacher's theorem leads to many more questions that can be answered.

Recently there has been a wealth of interest in generalising results of classical analysis to the setting of metric spaces, possibly with an underlying measure. Naturally, Rademacher's theorem is a candidate for such a generalisation. This course will focus on a new proof of Cheeger's generalisation which replaces the domain with a doubling metric measure space (X, d, μ) that satisfies a Poincaré inequality [Che99]. This proof uses modern techniques developed in [Bat14] which consider a rich structure of Lipschitz curves in the metric space (known as "Alberti representations") which allow us to form a partial derivative of any Lipschitz function $f\colon X \to \mathbb{R}$. By considering many such families of curves, we are able to form a derivative of f and hence deduce Cheeger's theorem.

This constitutes the third proof of Cheeger's theorem. In addition to the original, Keith [Kei04] gave an independent proof (see also [KM16]). One common element to all three of these proofs is the use of weak tangent spaces (defined via Gromov-Hausdorff convergence), which is only natural considering the nature of the derivative. Essentially, after an analysis of a large collection of Lipschitz functions in the metric space X, one takes a weak limit and the tangential behaviour of these Lipschitz functions is significantly more rigid (i.e. linear in the classical theory). The doubling condition then gives an upper bound on the number of different (i.e. linearly independent) tangential behaviours the large set of Lipschitz functions can posses, which ultimately leads to the existence of a derivative.

2. Analysis on metric spaces and the Poincaré inequality

Definition 2.1. A *metric measure space* (X, d, μ) consists of a complete metric space (X, d) equipped with a σ-finite Borel measure μ. That is, all Borel subsets of X are measurable and there exists a decomposition

$$X = \bigcup_{i \in \mathbb{N}} X_i$$

into Borel sets such that $\mu(X_i) < \infty$ for each $i \in \mathbb{N}$.

For many purposes, this definition includes spaces that are simply too large to work in. Often, it is natural to impose the following mild "finite dimensional" condition.

Given a metric space X, $x \in X$ and $r > 0$, we will denote by $B(x,r)$ the closed ball centred at x with radius r. Given a ball B and $\lambda > 0$, we will denote by λB the ball with the same centre and with radius λ times bigger.

Definition 2.2. A metric measure space (X, d, μ) is *doubling* if there exists a $C \geq 1$ such that, for each ball $B \subset X$,

$$0 < \mu(2B) \leq C\mu(B) < \infty.$$

We will write $\operatorname{doub}(\mu)$ for the least such C, the *doubling constant* of μ. In particular, by induction,

$$\mu(2^n B) \leq C^n \mu(B) \tag{2.1}$$

for each $n \in \mathbb{N}$.

An example of doubling metric measure spaces are *Ahlfors regular* metric measure spaces. A metric measure space (X, d, μ) is α-Ahlfors regular, for some $\alpha \geq 0$ if there exists a $C \geq 1$ such that

$$r^\alpha / C \leq \mu(B(x,r)) \leq C r^\alpha$$

for each $x \in X$ and $r > 0$.

Related to doubling measures are doubling metric spaces.

Definition 2.3. A metric space X is *doubling* if there exists an $N \in \mathbb{N}$ such that each ball $B \subset X$ is covered N balls of half the radius of B.

The smallest such N is called the *doubling constant* of X and is denoted by $\operatorname{doub}(X)$.

Lemma 2.4. *If (X, d, μ) is a doubling metric measure space then (X, d) is a doubling metric space. Moreover,* $\operatorname{doub}(X)$ *is bounded from above by a quantity depending only upon* $\operatorname{doub}(\mu)$.

Proof. Let $B \subset X$ be a ball of radius r and let $\mathcal{X}$ be a maximal $r/2$-net of B. That is, $d(x, x') \geq r/2$ for any $x, x' \in \mathcal{X}$ and, for any $y \in X$, there exists $x \in \mathcal{X}$ with $d(x, x') \leq r/2$. Thus

$$B \subset \bigcup_{x \in \mathcal{X}} B(x, r/2) \tag{2.2}$$

and

$$\bigcup_{x \in \mathcal{X}} B(x, r/4) \subset 2B \tag{2.3}$$

and this final union is disjoint.

We claim that the size of $\mathcal{X}$ is finite, and depends only upon $\operatorname{doub}(\mu)$. Indeed, suppose that there were at least n elements $x_1, \ldots, x_n \in \mathcal{X}$. Then by eq. (2.1), for each $1 \leq i \leq n$,

$$\mu(2B) \leq \mu(B(x_i, 4r)) \leq \operatorname{doub}(\mu)^4 \mu(B(x_i, r/4))$$

Thus, since the $B(x_i, r/4)$ are disjoint, eq. (2.3) gives

$$n\mu(2B) \leq \sum_{i=1}^{n} \operatorname{doub}(\mu)^4 \mu(B(x_i, r/4)) \leq \operatorname{doub}(\mu)^4 \mu(2B),$$

so that $n \leq \operatorname{doub}(\mu)^4$. Equation (2.2) completes the proof. □

Observation 2.5. *Let X be a doubling metric space. Then by induction, any ball $B(x,r)$ can be covered by* $\operatorname{doub}(X)^n$ *balls of radius $2^{-n}r$. In particular, for any $\alpha > \log_2 \operatorname{doub}(X)$, $\mathcal{H}^\alpha(B(x,r)) = 0$. Therefore,* $\dim_H X \leq \log_2 \operatorname{doub}(X)$.

A standard property of doubling measures is that they satisfy the Vitali covering theorem, and hence the Lebesgue density theorem. The proofs of these facts follow the classical proofs with minor superficial changes. See for example [**heinonen**].

Theorem 2.6 (Lebesgue differentiation theorem)**.** *Let (X,d,μ) be a doubling metric measure space and $f\colon X\to\mathbb{R}$ a positive integrable function. Then*

$$\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int_{B(x,r)} f\,\mathrm{d}\mu = f(x)$$

for μ-a.e. $x\in X$.

In particular, if $S\subset X$ is measurable, for μ-a.e. $x\in S$,

$$\lim_{r\to 0}\frac{\mu(B(x,r)\cap S)}{\mu(B(x,r))}=1. \tag{2.4}$$

Definition 2.7. Let X be a metric space and $S\subset X$. A point $x\in S$ is a *porosity point* of S if there exist an $\eta>0$ and $x_n\to x$ such that

$$B(x_n,\eta d(x,x_n))\cap S=\emptyset \tag{2.5}$$

for each $n\in\mathbb{N}$. Further, S is *porous* if all of its points are porosity points.

Corollary 2.8. *Let (X,d,μ) be a doubling metric measure space and $S\subset X$. Then the set of porosity points of S has μ measure zero.*

Proof. Suppose that $x\in S$ satisfies eq. (2.5) for some $0<\eta\le 1$. For each $n\in\mathbb{N}$, eq. (2.1) gives

$$\begin{aligned}\mu(B(x_n,\eta d(x,x_n))) &\ge \mathrm{doub}(\mu)^{-N}\mu(B(x_n,2d(x,x_n)))\\ &\ge \mathrm{doub}(\mu)^{-N}\mu(B(x,d(x,x_n))),\end{aligned}$$

for $N=[2/\eta]$. In particular,

$$\begin{aligned}\mu(B(x,2d(x,x_n))\cap S) &\le \mu(B(x,2d(x,x_n))\setminus B(x_n,\eta d(x,x_n)))\\ &\le (1-\mathrm{doub}(\mu)^{-N})\mu(B(x,2d(x,x_n))).\end{aligned}$$

Therefore, x does not satisfy eq. (2.4), and so the set of such x has μ measure zero. □

We introduce some standard notation. For an integrable $f\colon(X,d,\mu)\to\mathbb{R}$ and $B\subset X$ of positive and finite measure, let

$$\fint_B f\,\mathrm{d}\mu=\frac{1}{\mu(B)}\int_B f\,\mathrm{d}\mu$$

and $f_B=\fint_B f\,\mathrm{d}\mu$.

The proof of the classical Poincaré inequality relies on using the fundamental theorem of calculus along lines in the domain. This motivates our definition of a Poincaré inequality in the setting of metric measure spaces, but we must replace a "line" by a rectifiable curve. That is, a Lipschitz function $\gamma\colon[0,l]\to X$, for some $l\ge 0$. However, we need a concept to replace the derivative in the classical setting.

Recall that the *poitwise Lipschitz constant* of a Lipschitz function $f\colon(X,d)\to(Y,\rho)$ is defined by

$$\mathrm{Lip}(f,x):=\limsup_{y\to x}\frac{\rho(f(x),f(y))}{d(x,y)}.$$

Definition 2.9. Let $\gamma\colon[0,l]\to X$ be Lipschitz and $\rho\colon X\to\mathbb{R}$ be Borel. The *line integral* of ρ over γ is

$$\int_\gamma \rho\,\mathrm{d}s=\int_0^l \rho(\gamma(t))\,\mathrm{Lip}(\gamma,t)\,\mathrm{d}t.$$

We can now define a replacement for the fundamental theorem of calculus in a metric space.

Definition 2.10. Let X be a metric space and $f\colon X \to \mathbb{R}$ a Lipschitz function. A Borel function $\rho\colon X \to [0,\infty)$ is an *upper gradient* of f if, for every $x, y \in X$ and every rectifiable curve γ joining x to y,

$$|f(x) - f(y)| \leq \int_\gamma \rho \,\mathrm{d}s.$$

Observe that, if $X = \mathbb{R}^n$, then by Lebesgue's theorem, $\rho = \|Df\|$ is an upper gradient of any Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}$. More generally, for any metric space X and any Lipschitz $f\colon X \to \mathbb{R}$, $\mathrm{Lip}(f,\cdot)$ is an upper gradient of f. Indeed, this is immediate from the inequality

$$|(f\circ\gamma)'(t)| \leq \mathrm{Lip}(f,\gamma(t))\,\mathrm{Lip}(\gamma,t).$$

Observe that the upper gradient is only interesting when X has a rich structure of rectifiable curves. For example, if X contains no non-trivial rectifiable curves, as is the case when $X = \{0,1\}$, then $\rho = 0$ is an upper-gradient of *any* Lipschitz function.

The notion of a Poincaré inequality in a metric measure space says that any upper gradient of a Lipschitz function must control the behaviour in a very precise way. The Poincaré inequality in this setting was first introduced by Heinonen and Koskela [**Heinonen˙1998**]. We give a slightly different formulation that is equivalent whenever the measure is doubling.

Definition 2.11. For $p \geq 1$, a metric measure space (X,d,μ) satisfies a *p-Poincaré inequality* if there exists a $C \geq 1$ such that, for any ball $B \subset X$, any Lipschitz $f\colon X \to \mathbb{R}$ and any upper-gradient ρ of f,

$$\fint_B |f - f_B| \leq C\,\mathrm{rad}(B)\left(\fint_B \rho^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}. \tag{2.6}$$

We say that (X,d,μ) is a *p-PI space* if it is doubling and satisfies a p-Poincaré inequality. We will write $\mathrm{PI}(\mu)$ for the least $C \geq \mathrm{doub}(\mu)$ for which eq. (2.6) holds.

By Holder's inequality, the Poincaré inequality becomes weaker as p increases. The classical Poincaré inequality shows that that Euclidean space is a 1-PI space. The class of PI spaces is very rich (see [Kei04, p. 274]) and in particular include all limits of Riemannian manifolds whose Ricci curvatures are uniformly bounded from below and all Carnot groups. Laakso gave several very interesting, non-trivial, and highly non-Euclidean examples of 1-PI spaces [**laakso**; **laakso-graph**].

3. Rademacher's theorem for metric measure spaces: Cheeger's theorem and Alberti representations

We now move on to considering the derivatives of real valued Lipschitz functions defined on a metric space. The first way to generalise a derivative to this setting was introduced by Cheeger [Che99] and was later refined by Keith [Kei04]. It simply replaces to coordinate functions in Euclidean space by an arbitrary vector valued Lipschitz function.

Definition 3.1. Let $\phi\colon X \to \mathbb{R}^n$ be a fixed Lipschitz function and $x_0 \in X$.

A function $f\colon X \to \mathbb{R}$ is *differentiable with respect to* ϕ at x_0 if there exists a unique $Df(x_0) \in \mathbb{R}^n$ such that

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0) - Df(x_0)\cdot(\phi(x) - \phi(x_0))|}{d(x,x_0)} = 0.$$

Note that we must *assume* that the derivative is unique as a part of the definition.

We can now say what it means for a metric measure space to satisfy a generalisation of Rademacher's theorem.

Definition 3.2. A metric measure space (X, d, μ) is a *Lipschitz differentiability space* if there exists a countable Borel decomposition

$$X = \bigcup_{i \in \mathbb{N}} U_i$$

and countably many Lipschitz functions $\phi_i \colon X \to \mathbb{R}^{n_i}$ such that the following is true: For every Lipschitz $f \colon X \to \mathbb{R}$ and every $i \in \mathbb{N}$, f is differentiable at μ-a.e. point in U_i with respect to ϕ_i.

We call the pair (U_i, ϕ_i) a *chart.*

This is all the required concepts to state Cheeger's theorem [Che99].

Theorem 3.3 (Cheeger)**.** *Any PI space is a Lipschitz differentiability space.*

The fact that a function f is differentiable with respect to ϕ at a point x_0 should be compared to the statement that a vector v is a linear combination of a collection of vectors $p_1, p_2, \ldots, p_n$. Essentially, it says that the components of ϕ span all the possible infinitesimal behaviours of a Lipschitz function at x_0.

The fact that the derivative is unique is comparable to the fact that the components of ϕ are linearly independent. Indeed, since the zero function must have a unique derivative, namely derivative zero, at almost every point, we see that for almost every x_0, $\operatorname{Lip}(D \cdot \phi, x_0) = 0$ implies $D = 0$. It is easy to see that either of these two conditions are equivalent to the fact that the derivative of any Lipschitz function is unique almost everywhere.

This idea can be expanded upon to give the following characterisation of Lipschitz differentiability spaces. It is comparable to the following statement: Suppose that, for a vector space V, there exists a $N \in \mathbb{N}$ such that any collection of linearly independent vectors has size at most N. Then V has a basis of size at most N.

We require the following measure theoretic property: Suppose that μ is a σ-finite measure on X and that $\mathcal{T}$ is a collection of μ measurable sets such that any positive measure subset of X contains a positive measure element of $\mathcal{T}$. Then we can decompose almost all of X into a countable union of elements of $\mathcal{T}$.

Proposition 3.4. *Let (X, d, μ) be a σ-finite metric measure space and suppose that there exists an $N \in \mathbb{N}$ for which the following is true. When ever $\phi \colon X \to \mathbb{R}^n$ is Lipschitz with the property that*

$$\operatorname{Lip}(D \cdot \phi, x) > 0 \quad \textit{for every } D \in \mathbb{R}^n \setminus \{0\}, \tag{3.1}$$

for a set of positive measure $x \in X$, then $n \leq N$. Then (X, d, μ) is a Lipschitz differentiability space.

Proof. Let $U \subset X$ be a Borel set of positive measure. Either every Lipschitz $\phi \colon X \to \mathbb{R}$ satisfies $\operatorname{Lip}(\phi, x) = 0$ for μ almost every $x \in U$ or there exists a $U_1 \subset U$ of positive measure and a Lipschitz $\phi_1 \colon X \to \mathbb{R}$ with $\operatorname{Lip}(\phi_1, x) > 0$ for every $x \in U_1$. Given the first option we stop; in this case X is a Lipschitz differentiability space with respect to the zero function. Otherwise we proceed iteratively.

Suppose that we have, for some $n \in \mathbb{N}$, a Lipschitz $\phi_n \colon X \to \mathbb{R}^n$ and a $U_n \subset U$ with $\mu(U_n) > 0$ such that $\operatorname{Lip}(D \cdot \phi, x) > 0$ for every $D \in \mathbb{S}^{n-1}$ for almost every $x \in U_n$. Then either

- (U_n, ϕ) is a chart with respect to which every Lipschitz $f \colon X \to \mathbb{R}$ is differentiable almost everywhere;

- or there exists a $U_{n+1} \subset U_n$ of positive measure and a Lipschitz $\phi^{n+1} \colon X \to \mathbb{R}$ such that $\phi' = (\phi, \phi^{n+1})$ satisfies $\operatorname{Lip}(D \cdot \phi', x) > 0$ for every $D \in \mathbb{R}^{n+1} \setminus \{0\}$.

Note, in the first case, as mentioned in the discussing preceeding the proposition, the uniqueness of the derivative is equivalent to the induction hypotheses on ϕ.

By hypothesis, the second option cannot hold for sufficiently large n, and so at some point we find a subset of U of positive measure which forms a chart. The standard measure theory result completes the proof. □

We now introduce an alternative way to generalise Rademacher's theorem to metric measure spaces. However, it turns out that this in an equivalent formulation to Cheeger's. This is nice because it allows us to have two very different descriptions of the same phenomenon.

This alternative formulation is based on the idea of forming a *partial derivative* of any Lipschitz function along a rectifiable curve.

Definition 3.5. Given a metric space X, define the set of *curve fragments* in X to be

$$\Gamma(X) = \{\gamma \colon \operatorname{dom}\gamma \subset [0,1] \to X : \operatorname{dom}\gamma \text{ compact, non-empty, } \gamma \text{ 2-biLipschitz}\}.$$

Given $\gamma \in \Gamma(X)$ let

$$\operatorname{Graph}(\gamma) = \{(t, \gamma(t)) : t \in \operatorname{dom}\gamma\} \subset [0,1] \times X.$$

This is injective and so we can define a metric on $\Gamma(X)$ as the Hausdorff metric on the graphs of curve fragments.

We do not assume that $\operatorname{dom}\gamma$ is connected; one should imagine that $\operatorname{dom}\gamma$ is a fat Cantor set. This is important, so that $\Gamma(X)$ is much richer than the set of rectifiable *curves* in X. The details of th metric on $\Gamma(X)$ are not important to us, only the fact that its existence allows us to consider Borel measures on Γ.

Definition 3.6. An *Alberti representation* of a metric measure space (X, d, μ) consists of a probability measure $\mathbb{P}$ on $\Gamma(X)$ and for each $\gamma \in \Gamma(X)$ a measure $\mu_\gamma \ll \mathcal{H}^1|_\gamma$ such that

$$\mu(B) = \int_{\Gamma(X)} \mu_\gamma(B) \, \mathrm{d}\mathbb{P}(\gamma)$$

for every Borel $B \subset X$.

The motivation for considering Alberti representations is the following. Suppose that $f \colon X \to \mathbb{R}$ is Lipschitz and that $\gamma \in \Gamma(X)$. Then the composition

$$f \circ \gamma \colon \operatorname{dom}\gamma \to \mathbb{R}$$

is Lipschitz and so is differentiable almost everywhere by Lebesgue's theorem. That is, for $\mathcal{H}^1$ almost every $x \in \operatorname{Im}\gamma$, there exists a *partial derivative* of f at x given by $(f \circ \gamma)'(\gamma^{-1}(x))$. Equivalently, the set S of points for which no such partial derivative exists satisfies $\mathcal{H}^1(\gamma \cap S) = 0$, and hence $\mu_\gamma(S) = 0$, for each $\gamma \in \Gamma(X)$. Thus, if (X, d, μ) has an Alberti representation, such a partial derivative exists for μ almost every $x \in X$.

This should be compared to a standard proof of Rademacher's theorem, where a first step is to use the Fubini and Lebesgue theorems to find a partial derivative of any Lipschitz function. Here, we are replacing the use of Fubini's theorem with the *definition* of an Alberti representation.

Of course, in the proof of Rademacher's theorem, it is important that we can find all n partial derivatives of a Lipschitz $f \colon \mathbb{R}^n \to \mathbb{R}$. Our next step is to interpret this for Alberti representations.

In Euclidean space, we can distinguish different families of curves by considering their tangents: some curves travel in a direction close to the e_1 direction, some others close to the e_2 direction and so on. In metric spaces there is no natural way to assign a tangent. However, we can use a Lipschitz $\phi\colon X \to \mathbb{R}^n$ to pull back the geometry of $\mathbb{R}^n$ to the metric space.

Definition 3.7. Let (X, d) be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. We say that $\gamma \in \Gamma(X)$ is a C-curve (with respect to ϕ) if

$$\phi(t) - \phi(s) \in C \setminus \{0\} \quad \text{for every } s, t \in \operatorname{dom}\gamma.$$

Similarly, an Alberti representation $(\mathbb{P}, \{\mu_\gamma\})$ of a metric measure space is a C-Alberti representation (with respect to ϕ) if $\mathbb{P}$ almost every curve is a C curve (with respect to ϕ).

Finally, we say that a collection of Alberti representations

$$(\mathbb{P}_1, \{\mu_\gamma^1\}), \ldots, (\mathbb{P}_n, \{\mu_\gamma^n\})$$

is *independent* if there exists a Lipschitz $\phi\colon X \to \mathbb{R}^n$ and independent cones $C_1, \ldots, C_n \subset \mathbb{R}^n$ such that $(\mathbb{P}_i, \{\mu_\gamma^i\})$ is a C_i-Alberti representation with respect to ϕ for each $1 \le i \le n$.

We say that cones $C_1, \ldots, C_n \subset \mathbb{R}^n$ are *independent* if any choice $v_i \in C_i \setminus \{0\}$ forms a linearly independent set. Often the underlying function ϕ will be clear from the context and we will not explicitly mention it.

We see from Fubini's theorem that Lebesgue measure has a collection of n independent Alberti representations (and no more), and that this is precisely the property required in the proof of Rademacher's theorem.

The first half of our proof of Cheeger's theorem will be to prove that a PI space has a large collection of independent Alberti representations. Before we prove this, we will develop a general technique for deciding when a measure has many independent Alberti representations. To begin, we state a result characterising the existence of a single Alberti representation of a measure.

Definition 3.8. Let X be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. A Borel $S \subset X$ is *C-null (with respect to ϕ)* if $\mathcal{H}^1(\gamma \cap S) = 0$ for each C-curve γ.

Observe that, if a metric measure space is C-null then it cannot support a C-Alberti representation. This is in fact a characterisation of those measures with C-Alberti representations. We will use this fact without proof.

Proposition 3.9. *Let (X, d, μ) be a metric measure space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. There exists a decomposition $X = A \cup S$ such that $\mu \llcorner A$ has a C-Alberti representation and S is C-null. This decomposition is unique up to μ null sets.*

This result follows from a generalised Lebesgue decomposition theorem. Observe that, for any $\gamma \in \Gamma(X)$, the Lebesgue decomposition theorem given a decomposition $X = A \cup S$ such that $\mu \llcorner A \ll \mathcal{H}^1|_\gamma$ and $\mathcal{H}^1|_\gamma(S) = 0$. The mentioned generalised Lebesgue decomposition theorem allows us to do this simultaneously for all C-curves. See [Bat14, Section 5.1].

By applying the previous Proposition multiple times using a collection of independent cones $C_1, C_2, \ldots, C_n$, we obtain a decomposition $X = A \cup S_1 \cup S_2 \cup \ldots \cup S_n$ where each S_i is C_i-null and $\mu|_A$ has n independent Alberti representations. Thus, we must find suitable conditions on μ to imply that any C_i-null subsets have measure zero. We will see that for PI spaces this is the case whenever the C_i are sufficiently wide. Of course, we cannot simply apply the Proposition to very wide cones, because they will not be independent. Therefore, we must do something more involved.

4. Alberti representations from the Poincaré inequality

We begin with some geometric results regarding Alberti representations.

Lemma 4.1. *Let X be a metric space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. Suppose that $S \subset X$ is C-null with respect to ϕ. Then for any $\gamma \in \Gamma(X)$,*

$$(\phi \circ \gamma)'(t) \notin \operatorname{interior}(C)$$

for almost every $t \in \gamma^{-1}(S)$.

Proof. Suppose that the conclusion is false and that for some $\gamma \in \Gamma(X)$,

$$B = \{t \in \gamma^{-1}(S) : (\phi \circ \gamma)'(t) \in \operatorname{interior}(C)\}$$

has positive measure. Let

$$\operatorname{interior}(C) = \bigcup_{i \in \mathbb{N}} C_i$$

where each C_i is closed and convex. Then there exists some $i \in \mathbb{N}$ such that

$$B_1 = \{t \in B : (\phi \circ \gamma)'(t) \in C_i\}$$

has positive measure. Further, since C_i is a closed subset of interior(C), there exists an $R > 0$ such that the set

$$B_2 = \{t \in B_1 : \frac{\phi(\gamma(t+r)) - \phi(\gamma(t))}{r} \in C \setminus \{0\} \ \forall\ 0 < |r| < R\}$$

also has positive measure.

By dividing B_2 into finitely many subsets of diameter R, we can find one such subset B_3 of positive measure. In particular, for any $s, t \in B_3$,

$$\frac{\phi(\gamma(t)) - \phi(\gamma(s))}{t - s} \in C \setminus \{0\}.$$

Finally, by taking a compact $K \subset B_3$ of positive measure, we see that $\gamma|_K$ is a C-curve intersecting S in a set of positive measure. This is a contradiction. □

The previous lemma gives us the following method to "refine" the directions of an Alberti representation. Of course, this makes most sense when the C_i are very thin.

Lemma 4.2. *Let X be a metric space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. Suppose that a measure μ has a C-Alberti representation with respect to ϕ. Then, for any collection of cones $C_1, \ldots, C_m \subset \mathbb{R}^n$ such that*

$$C \setminus \{0\} \subset \bigcup_{i=1}^{m} \operatorname{interior}(C_i),$$

there exists a Borel decomposition $X = X_1 \cup \ldots \cup X_m$ such that each $\mu|_{X_i}$ has a C_i-Alberti representation.

Proof. By Proposition 3.9 there exists a decomposition $X = X_1 \cup S_1$ such that X_1 has the required form and S_1 is C_1-null. By applying Proposition 3.9 again (to $\mu|_{S_1}$) we obtain a decomposition $X = X_1 \cup X_2 \cup S_2$ where X_2 has the required form and S_2 is both C_1-null and C_2-null. By repeating, we obtain a decomposition $X = X_1 \cup \ldots \cup X_m \cup S$ such that each X_i has the required form and S is C_i-null for each $1 \leq i \leq m$. Therefore, by Lemma 4.1, S is C-null. Since μ has a C-Alberti representation, we must have $\mu(S) = 0$. Therefore, $X_1' = X_1 \cup S$ also has the required properties of X_1, and $X = X_1' \cup X_2 \cup \ldots \cup X_m$ is the required decomposition. □

Finally, we may obtain the required decomposition mentioned at the end of Section 3.

Definition 4.3. For $w \in \mathbb{S}^{n-1}$ and $0 < \theta < 90$, let $C(w,\theta) \subset \mathbb{R}^n$ be the cone centred on w with interior angle θ.

Let X be a metric space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. Define the set $\tilde{A}(\phi)$ to be the collection of $S \subset X$ for which the following is true: For any $0 < \theta < 1$ there exists a Borel decomposition

$$S = S_1 \cup S_2 \cup \ldots \cup S_m$$

and $w_1, \ldots, w_m \in \mathbb{S}^{n-1}$ such that S_i is $C(w_i, 90-\theta)$-null for each $1 \leq i \leq m$.

Theorem 4.4. *Let (X,d,μ) be a metric measure space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. There exists a Borel decomposition*

$$X = S \cup \bigcup_{i\in\mathbb{N}} A_i$$

such that each $\mu|_{A_i}$ has n independent Alberti representations and $S \in \tilde{A}(\phi)$.

Proof. We first prove the following. Given any $0 < \theta < 1$ and $1 \leq d \leq n$ there exists a Borel decomposition

$$X = \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{m} S_i \tag{4.1}$$

such that each $\mu|_{A_i}$ has d independent Alberti representations and each S_i is C_i-null, for some C_i with interior angle $90-\theta$. We iteratively find these Alberti representations one-by-one as follows.

First apply Proposition 3.9 using an arbitrary cone C of interior angle $90-\theta$ to obtain a decomposition $X = A \cup S$ where $\mu|_A$ has a C-Alberti representation and S is C-null.

Now suppose that, for $1 \leq d < n$ we have a decomposition as in eq. (4.1) where each $\mu|_{A_i}$ has d independent Alberti representations. For a $0 < \alpha < 1$ to be determined later, by applying Lemma 4.2 (and increasing m) we may suppose that there exists $w_1^i, \ldots, w_d^i \in \mathbb{S}^{n-1}$ such that these Alberti representations are $C(w_j^i, \alpha)$-Alberti representations.

Fix a $1 \leq i \leq m$ and pick w_{d+1}^i to be orthogonal to $w_1^i, \ldots, w_d^i$, which is possible because, by assumption, $1 \leq d < n$. We pick α so small that $C(w_{d+1}^i, 90-\theta)$ is disjoint from each $C(w_j^i, \alpha)$ (this is independent of the choice of the w_j^i and only depends on n). Then the cones $C(w_1^i, \alpha), \ldots, C(w_d^i, \alpha)$ and $C(w_{d+1}^i, \theta)$ are independent. Now apply Proposition 3.9 to obtain a decomposition $A_i = A_i' \cup S'$ where $\mu|_{A_i'}$ has $d+1$ independent Alberti representations and S' is $C(w_{d+1}^i, 90-\theta)$-null.

Repeating this for each $1 \leq d < n$ completes the proof of eq. (4.1). To complete the proof of the Theorem we apply eq. (4.1) for each $j \in \mathbb{N}$ with $\theta = 1/j$ to obtain a decomposition $X = \hat{A}_j \cup \hat{S}_j$ where each $\hat{A}_j$ is a finite union of sets with n independent Alberti representations and each $\hat{S}_j$ has a finite decomposition into sets that are C-null for some cone C of interior angle $90-1/j$. Setting $S = \cap_j \hat{S}_j$ completes the proof. □

Given a Lipschitz $\phi\colon X \to \mathbb{R}^n$, we a way to produce many independent Alberti representations (and hence many partial derivatives of any Lipschitz function) after excluding a certain $\tilde{A}(\phi)$ set. The next step is to show that we can remove this $\tilde{A}(\phi)$ set when considering PI spaces. Of course, we cannot deduce that there arbitrarily large collections of independent Alberti representations; $\mathbb{R}^n$ can have at most n

independent Alberti representations. Observe that the main obstacle here is, for poorly chosen ϕ, the whole of X is a $\tilde{A}(\phi)$ set. For example, consider what happens when $\phi = 0$, or $X = \mathbb{R}$ and $\phi(x) = (x, x)$. The problem is that the components of ϕ are linearly dependent.

We have seen this condition before when considering the uniqueness of Cheeger's derivative. We will now show that for ϕ satisfying eq. (3.1), $\tilde{A}(\phi)$ subsets of PI spaces have measure zero.

Lemma 4.5. *Let (X, d, μ) be a doubling metric measure space and $f \colon X \to \mathbb{R}$ 1-Lipschitz. There exists a constant $\eta = \eta(\mathrm{doub}(\mu)) \geq 1$ such that, for any ball $B \subset X$,*

$$\fint_{2B} |f - f_{2B}| \,\mathrm{d}\mu \geq C(\mu) \operatorname{rad} B \sup \left\{ \frac{|f(x) - f(y)|}{\operatorname{rad} B} : x, y \in B \right\}^{\eta}.$$

In particular, if (X, d, μ) is a PI space and ρ is an upper gradient of f,

$$\operatorname{Lip}(f, x) \leq C(\mathrm{PI}(\mu)) \rho(x)^{1/\eta}$$

for μ almost every $x \in X$.

Remark 4.6. The 'in particular' statement of the previous lemma is the only part in the whole proof Cheeger's theorem that requires the Poincaré inequality.

Proof. Fix a ball $B \subset X$ and let

$$M = \sup \left\{ \frac{|f(x) - f(y)|}{\operatorname{rad} B} : x, y \in B \right\} \leq 1$$

If $M = 0$ there is nothing to prove and so we may suppose $M > 0$. Since B is compact, there exist $x_1, x_2 \in B$ such that

$$|f(x_1) - f(y_2)| = M \operatorname{rad} B.$$

For $i = 1, 2$ let $B_i = B(x_i, M \operatorname{rad} B/10) \subset 2B$. Then, since f is 1-Lipschitz,

$$|f(y_1) - f(y_2)| \geq 8M \operatorname{rad} B/10$$

for each $y_i \in B_i$ and each $i = 1, 2$. In particular, there exists $i \in 1, 2$ such that

$$|f(y_i) - f_{2B}| \geq 4M \operatorname{rad} B/10$$

for each $y_i \in B_i$. Therefore,

$$\fint_{2B} |f - f_{2B}| \,\mathrm{d}\mu \geq \frac{1}{\mu(2B)} \int_{B_i} |f - f_{2B}| \,\mathrm{d}\mu \geq \frac{2M \operatorname{rad} B \mu(B_i)}{5\mu(2B)} \tag{4.2}$$

Now, $\operatorname{rad} B_i = M \operatorname{rad} B/10$ and so, since μ is doubling,

$$C(\mu)^n \mu(B_i) \geq \mu(2B)$$

for $n = -\log_2 M/10 + 2$. That is, if $\delta = \log_2 C > 0$ depending on $C(\mu)$ such that

$$\frac{\mu(B_i)}{\mu(2B)} \geq CM^{\delta}$$

Combining this with eq. (4.2) completes the first part of the proof.

The second part of the lemma simply follows from the Lebesgue differentiation theorem. □

Proposition 4.7. *Let (X, d, μ) be a PI space and $\phi \colon X \to \mathbb{R}^n$ Lipschitz satisfying eq. (3.1) for all $x \in U \subset X$. Then any $\tilde{A}(\phi)$ subset of U has μ measure zero.*

Proof. Since the definition of an $\tilde{A}(\phi)$ set is invariant under scaling ϕ, we may suppose that ϕ is 1-Lipschitz.

For each $x \in U$, the function

$$D \in \mathbb{S}^{n-1} \mapsto \operatorname{Lip}(D \cdot \phi, x)$$

is Lipschitz and positive by assumption. Therefore, it is bounded below by some $\lambda_x > 0$ for each $x \in U$. The map $x \mapsto \lambda_x$ is Borel and so we may decompose U into sets $U_i = \{x : \lambda_x > 1/i\}$, $i \in \mathbb{N}$. Since it suffices to prove that any $\tilde{A}(\phi)$ subset of each U_i has μ measure zero, we may suppose that $U = U_i$ for some $i \in \mathbb{N}$ and set $\lambda = 1/i$.

For $0 < \theta < 1$ and $w \in \mathbb{S}^{n-1}$, fix a $C(w, 90 - \theta)$-null set $S \subset U$ and set $f = w \cdot \phi \colon X \to \mathbb{R}$. We claim that

$$\rho = 1_{S^c} + \theta 1_S$$

is an upper gradient of f. Indeed, if $\gamma \colon [0, l] \to X$ is parametrised by arc-length, then by Lemma 4.1, $(\phi \circ \gamma)' \notin \operatorname{int}(C(w, \theta))$ for almost every $t \in \gamma^{-1}(S)$. That is,

$$|(f \circ \gamma)'(t)| = |w \cdot (\phi \circ \gamma)'(t)| \leq \theta \|(\phi \circ \gamma)'(t)\| \leq \theta \operatorname{Lip} \phi = \theta$$

for almost every $t \in \gamma^{-1}(S)$. Therefore, by the fundamental theorem of calculus,

$$|f(\gamma(l)) - f(\gamma(0))| \leq \int_0^l |(f \circ \gamma)'(t)| \leq \int_{\gamma^{-1}(S)} \theta + \int_{\gamma^{-1}(X \setminus S)} 1 = \int_0^l \rho(t) \, \mathrm{d}t,$$

as required.

By applying Lemma 4.5, we find a $C, \eta \geq 1$ depending only on $\mathrm{PI}(\mu)$ such that, for μ almost every $x \in S$,

$$0 < \lambda \leq \operatorname{Lip}(f, x) \leq C \rho(x)^{1/\eta} = C \theta^{1/\eta}.$$

This is impossible if θ is sufficiently small, and so we must have $\mu(S) = 0$. Precisely, any $C(w, \lambda^\eta / C)$-null subset of U has μ measure zero. Therefore, any $\tilde{A}(\phi)$ subset of U has μ measure zero, as required. □

We complete this section by summarising the main result.

Theorem 4.8. *Let (X, d, μ) be a PI space. Suppose that $U \subset X$ and $\phi \colon X \to \mathbb{R}^n$ is Lipschitz such that eq.* (3.1) *holds for all $x \in U$. Then there exists a countable Borel decomposition $U = \cup_i A_i$ such that each $\mu|_{A_i}$ has n independent Alberti representations.*

In particular, if there exists an $N \in \mathbb{N}$ such that, for any $U \subset X$ with $\mu(U) > 0$, $\mu|_U$ can have at most N independent Alberti representations, then X is a Lipschitz differentiability space.

Proof. The first part follows by combining Theorem 4.4 and Proposition 4.7.

By combining the hypotheses of the second part with the first part of the theorem, we precisely satisfy the hypotheses of Proposition 3.4. The conclusion follows. □

The last step of the proof is to use the doubling condition to find the N appearing in the hypotheses of the previous theorem.

5. Gromov–Hausdorff convergence, weak tangent spaces, and concluding the proof of Cheeger's theorem

We want to consider a notion of convergence of metric spaces. There are many variations of a core idea developed by Gromov, and are all referred to by "Gromov–Hausdorff convergence". The specific variation that allows us to define a tangent of a doubling metric space requires us to consider a distinguished point of the metric space. A *pointed metric space* (X,d,x) consists of a metric space (X,d) and a point $x \in X$. We will also consider the limiting behaviour of a Lipschitz function defined on the space. A *space-function* (X,d,x,ϕ) consists of a doubling pointed metric space (X,d,x) and a Lipschitz $\phi\colon X \to \mathbb{R}^n$.

Definition 5.1 (Gromov–Hausdorff)**.** A sequence (X_n,d_n,x_n) of pointed metric spaces *Gromov–Hausdorff converges* to a pointed metric space (X,d,x) if there exists a sequence of maps (called *Hausdorff approximations*) $\iota_n\colon X \to X_n$ with $\iota_n(x)=x_n$ such that, for any $R>0$,

$$d_n(\iota_n(y),\iota_n(z)) \to d(y,z) \tag{5.1}$$

uniformly on $B(x,R)$ and, for any $\epsilon>0$ and sufficiently large n,

$$B(\iota_n(B(x,R)),\epsilon) \supset B(x_i,R-\epsilon). \tag{5.2}$$

Further, a sequence of space-functions (X_n,d_n,x_n,ϕ_n) *Gromov–Hausdorff tangent* to a space function (X,d,x,ϕ) if $(X_n,d_n,x_n)\to(X,d,x)$ via Hausdorff approximations $\iota_n\colon X\to X_n$ and

$$\phi_n(\iota_n(z)) \to \phi(z) \tag{5.3}$$

for each $z\in X$.

Remark 5.2. The first condition says that the Hausdorff approximations become more and more isometric on compact sets, whilst the second says they become more and more surjective.

Using similar techniques to the Arzelà–Ascoli theorem, we have a compactness theorem for Gromov–Hausdorff convergence.

Theorem 5.3. *Let $N,L\geq 0$ and, for each $n\in\mathbb{N}$, let (X_n,d_n) be a doubling metric spaces with* $\operatorname{doub}(X_n)\leq N$, *$x_n\in X_n$ and $\phi_n\colon X\to\mathbb{R}^m$ be an L-Lipschitz function. Then there exists a doubling pointed metric space (X,d) (with* $\operatorname{doub}(X)\leq N$*), $x\in X$ and an L-Lipschitz $\phi\colon X\to\mathbb{R}^m$ such that, after possibly taking a subsequence, $(X_n,d_n,x_n,\phi_n)\to(X,d,x,\phi)$.*

Proof. We will only demonstrate the construction of the limit space (X,d,x). The existence of ϕ then follows from a superficial modification of the proof of the Arzelà–Ascoli theorem.

Fix $R>0$. For each $k,n\in\mathbb{N}$ let

$$Y_k^n=\{y_0^n,y_1^n,\ldots,y_{N(k)}^n\}$$

be a $1/k$-net of $B(x_n,R)$. Note that since the X_n are uniformly doubling, such a net, where the number of elements is independent of n, exists. We may also require that $y_0^n=x_n$ for each $n\in\mathbb{N}$. Note that, by our choice of notation, we are also requiring that the $1/k$-nets of each X_n are nested as k increases.

Consider the tuples

$$D_k^n := (d_n(y_i^n,y_j^n) : 1\leq i,j\leq N(k)) \in [0,R]^{N(k)^2}.$$

By the compactness of $[0,R]$, by taking a subsequence if necessary, we may suppose that there exists a $D_k\in[0,R]^{N(k)^2}$ such that $D_k^n\to D_k$ as $n\to\infty$. Let $\hat{B}_k$ be an arbitrary set of $N(k)$ elements, say

$$\hat{B}_k=\{x_1,\ldots,x_{N(k)}\}$$

and define $\hat{d}_k(x_i,x_j)=(D_k)_{i,j}$, so that

$$\hat{d}_k(x_i,x_j)=\lim_{n\to\infty} d_n(y_i^n,y_j^n). \tag{5.4}$$

Thus, $\hat{d}_k$ defines a pseudo metric on $\hat{B}_k$.

We now repeat this for each $k \in \mathbb{N}$ and take a diagonal subsequence such that eq. (5.4) holds for each $k \in \mathbb{N}$. Since the D_k^n are nested, the $\hat{B}_k$ are also nested and $\hat{d}_k|_{\hat{X}_j} = \hat{d}_j$ whenever $j \leq k$. Thus we may define $\hat{B} = \cup_k \hat{B}_k$ and $\hat{d} = \lim \hat{d}_k$ on $\hat{B}$, so that $\hat{d}$ is also a pseudo metric. We let $(\tilde{B}, \tilde{d})$ be the quotient metric space of $(\hat{B}, \hat{d})$. and let (B, d) be the completion of $(\tilde{B}, \tilde{d})$.

Now observe that, for any $n \in \mathbb{N}$ and $k \leq j \in \mathbb{N}$, D_k^n is a $1/k$-net of D_j^n. It follows that $\hat{B}_k$ is a $1/k$-net of $\hat{B}_j$ and hence of $\hat{B}$. Thus

$$N_k := \{[x] : x \in \hat{B}_k\}$$

is a $1/k$-net of $\tilde{B}$ and hence of B. We define $\iota_n \colon N_n \to B_n$ as follows. For any $y \in N_n$, choose the smallest $1 \leq i \leq N(k)$ such that $z = [x_i]$ and define $\iota_n(z) = y_i^n$. We extend ι_n to B by mapping any $z \in B$ to an arbitrarily chosen $\iota_n(x)$ with $x \in N_n$ and $x' \in B(x, 1/n)$.

In this construction, Y_k^n is a $1/k$-net of $B(x_n, R)$. It follows that $B = B(x_0, R)$. The final step of the construction is to take a final diagonal convergent subsequence as $R \to \infty$.

It is now a matter of checking the conclusion of the theorem is satisfied. For simplicity, we use the notation for $B = B(x, R)$ as above. Since

$$\iota_n(B(x, R)) \supset \iota_n(N_n) = B_n,$$

and B_n is a $1/n$-net of $B(x_n, R)$, we have

$$B(\iota_n(B(x_0, R)), 1/n) \supset B(x_n, R),$$

and hence eq. (5.2). To see eq. (5.1), observe that for any $k \in \mathbb{N}$ and $x, y \in N_k$,

$$d_n(\iota_n(x), \iota_n(y)) \to d(x, y)$$

by the definition of d. Since N_k contains only finitely many points, we can ensure that

$$|d_n(\iota_n(x), \iota_n(y)) - d(x, y)| < 1/k$$

for all $x, y \in N_k$ and all n greater than some N. Therefore, for any $x', y' \in B$ and $n \geq N$, there exists $x, y \in N_k$ with $d(x, x'), d(y, y') \leq 1/k$ and $\iota_n(x) = \iota_n(x')$ and $\iota_n(y) = \iota_n(y')$. Thus

$$\begin{aligned} |d_n(\iota_n(x'), \iota_n(y')) - d(x', y')| &\leq |d_n(\iota_n(x), \iota_n(y)) - d(x, y)| + |d(x, y) - d(x', y')| \\ &\leq 1/k + 2/k. \end{aligned}$$

□

In what follows, we will not need to consider the specific metric in a metric space (X, d) and so will simply refer to it as X. Given $\lambda > 0$, we will write λX for the metric space $(X, \lambda d)$.

Definition 5.4. Let (X, x) be a pointed metric space. A *Gromov–Hausdorff tangent* of X at x is any limit of a sequence of the form $(\lambda X, x)$ for $\lambda \to \infty$. We denote by $\mathrm{Tan}(X, x)$ the set of all Gromov–Hausdorff tangents of X at x.

Similarly, let (X, x, ϕ) be a space-function. A *Gromov–Hausdorff tangent* of (X, x, ϕ) is any limit of a sequence of the form $(\lambda X, x, \lambda(\phi - \phi(x)))$ for $\lambda \to \infty$. The set of all Gromov–Hausdorff tangents of (X, x, ϕ) is denoted by $\mathrm{Tan}(X, x, \phi)$.

If X is doubling, then for any $\lambda > 0$, λX is also doubling with the same constant. Thus, for any $x \in X$ and $\lambda_i \to \infty$, by the previous compactness theorem, there exists a subsequence λ_{i_k} such that $(\lambda_{i_k} X, x)$ converges to some tangent space. Therefore, $\mathrm{Tan}(X, x) \neq \emptyset$. Further, for any $\lambda > 0$ and any Lipschitz $\phi \colon X \to \mathbb{R}^n$, the function $\lambda\phi \colon \lambda X \to \mathbb{R}^n$ has the same Lipschitz constant as ϕ. Therefore, also by the previous theorem, $\mathrm{Tan}(X, x, \phi) \neq \emptyset$.

Note, however, that tangents are rarely unique. Certainly, if $(Y,y)\in \operatorname{Tan}(X,x)$ then also $(\lambda Y,y)\in \operatorname{Tan}(X,x)$ for any $\lambda>0$. However, much more non uniqueness properties can occur: for n_j an increasing sequence of integers, consider the tangents to the graph of the following function at the origin:

$$f(x)=x\chi_A$$

for

$$A=\{x\in\mathbb{R}:\exp(\exp(-n_j))\leq x\leq \exp(\exp(-n_{j-1})),\ j \text{ odd}\}.$$

Of course, this is just a simple example.

We will use the following fact.

Fact. *Gromov–Hausdorff convergence of space-functions is metrisable. That is, there exists a metric d_{GH} on the set $\mathcal{M}$ of all space-functions (which, by definition, consist of doubling metric spaces and Lipschitz functions) such that (X_n,x_n,ϕ_n) converges to (X,x,ϕ) if and only if*

$$d_{GH}((X_n,x_n,\phi_n),(X,x,\phi))\to 0.$$

By Theorem 5.3, $(\mathcal{M},d_{GH})$ is a countable union of compact metric spaces and hence is separable.

We now establish two results regarding tangent spaces that are adaptations of classical results of Preiss [**preiss-density**].

Recall from Definition 2.7, that $x\in A\subset X$ is not a porosity point of A if

$$\lim_{R\to 0}\sup_{y\in B(a,R)}\frac{d(y,A\cap B(a,R))}{R}=0. \tag{5.5}$$

Lemma 5.5 ([**MR2865538**], Proposition 3.1 and [Dav15], Lemma 3.2)**.** *Let X be a metric space and $\phi\colon X\to\mathbb{R}^m$ Lipschitz. Suppose that $A\subset X$ and that $a\in A$ is not a porosity point of A. Then* $\operatorname{Tan}(A,a,\phi)=\operatorname{Tan}(X,a,\phi)$.

Proof. Let $(Y,t)\in\operatorname{Tan}(A,a)$. Then there exists $\lambda_i\to\infty$ and Hausdorff approximations $\iota_n\colon(Y,y)\to(\lambda_nA,a)$. We define Hausdorff approximations, also called ι_n, into (λ_nX,a) by simply post composing with the inclusion of A into X. Certainly eqs. (5.1) and (5.3) remain true for these Hausdorff approximations, we just need to check eq. (5.2). Given $R,\epsilon>0$, since eq. (5.2) applies for the Hausdorff approximations into (λ_nA,a), there exists $N_1\in\mathbb{N}$ such that, for all $n\geq N_1$,

$$B(\iota_n(B(y,R)),\epsilon)\supset B(a,(R-\epsilon)/\lambda_n)\cap A.$$

By eq. (5.5), there exists $N_2\in\mathbb{N}$ such that, for all $n\geq N_2$,

$$B(B(a,(R-\epsilon)/\lambda_n)\cap A,\epsilon/\lambda_n)\supset B(a,(R-\epsilon)/\lambda_n).$$

Therefore, for all $n\geq N_1,N_2$,

$$B(\iota_n(B(t,R)),2\epsilon)\supset B(a,(R-2\epsilon)/\lambda_n),$$

as required.

Now suppose that $(Y,y)\in\operatorname{Tan}(X,a)$. Then there exists $\lambda_n\to 0$ and Hausdorff approximations $\iota_n\colon(Y,y)\to(\lambda_nX,a)$. We define Hausdorff approximations $\tilde\iota_n\colon(Y,y)\to(\lambda_nA,a)$ as follows. For $z\in Y$, if $\iota_n(z)\in A$ let $\tilde\iota_n(z)=\iota_n(z)$. Otherwise, pick $p\in A$ with

$$d(\iota_n(z),p)\leq d(\iota_n(z),A)+\frac{1}{n\lambda_n} \tag{5.6}$$

and set $\tilde\iota_n(z)=p$. Note that eq. (5.2) is automatically true for $\tilde\iota_n$. Observe that, for any $R>0$, eqs. (5.5) and (5.6) show that

$$\lambda_n d(\iota_n(z),\tilde\iota_n(z))\to 0$$

uniformly on $B(t,R)$. Therefore, eqs. (5.1) and (5.3) for the $\tilde{\iota}_n$ follow from the corresponding properties for the ι_n, using the fact that the $\lambda_n\phi$ are uniformly Lipschitz on $\lambda_n X$ for eq. (5.3). □

The previous observation allows us to show that moving the base point of a tangent gives a space that is also a tangent.

Theorem 5.6 ([**MR2865538**], Theorem 1.1 and [Dav15], Proposition 3.1)**.** *Let (X,μ) be a doubling metric measure space and $\phi\colon X\to\mathbb{R}^m$ Lipschitz. Then for μ almost every $x\in X$ the following is true. If $(Y,y,\psi)\in\operatorname{Tan}(X,x,\phi)$ and $y'\in Y$, then $(Y,y',\psi-\psi(y'))\in\operatorname{Tan}(X,x,\phi)$.*

Proof. We must show that the following set has μ measure zero:

$$\{x\in X : \exists\ (Y,y,\psi)\in\operatorname{Tan}(X,x,\phi),\ y'\in Y \text{ s.t. } (Y,y',\psi-\psi(y'))\notin\operatorname{Tan}(X,x,\phi)\}.$$

If $(Y,y',\psi-\psi(y'))\notin\operatorname{Tan}(X,x,\phi)$ then there exists $k\in\mathbb{N}$ such that

$$d((Y,y',\psi-\psi(y')),(\lambda X,x,\lambda(\phi-\phi(x)))>1/k \quad \forall\ \lambda>k.$$

Thus, it suffices to prove, for a $\delta>0$ which we now fix, that the following set has μ measure zero:

$$\{x\in X : \exists\ (Y,y,\psi)\in\operatorname{Tan}(X,x,\phi),\ y'\in Y \text{ s.t.}\\ d((Y,y',\psi-\psi(y')),(\lambda X,x,\phi))>\delta\ \forall\ \lambda>\delta\}.$$

Since the set of all space-functions is separable with respect to Gromov–Hausdorff convergence, there exist a countable cover by sets of diameter $\delta/4$. Let B be one such set. It suffices to prove that the set A defined to be those $x\in X$ for which

$$\exists\ (Y,y,\psi)\in\operatorname{Tan}(X,x,\phi),\ y'\in Y \text{ s.t. } (Y,y',\psi-\psi(y'))\in B \text{ and}\\ d((Y,y',\psi-\psi(y')),(\lambda X,x,\phi))>\delta\ \forall\ \lambda>1/\delta$$

has measure zero. Finally, by Corollary 2.8 and Lemma 5.5, for μ almost every $x\in A$, $\operatorname{Tan}(X,x,\phi)=\operatorname{Tan}(A,x,\phi)$. Therefore, it suffices to prove that the set A' of those $x\in A$ for which

$$\exists\ (Y,y,\psi)\in\operatorname{Tan}(A,x,\phi),\ y'\in Y \text{ s.t. } (Y,y',\psi-\psi(y'))\in B \text{ and}\\ d((Y,y',\psi-\psi(y')),(\lambda X,x,\phi))>\delta\ \forall\ \lambda>1/\delta$$

has measure zero. In fact, we will show that it is empty.

Indeed, suppose that $x\in A'$. Then there exist $\lambda_n\to\infty$ and Hausdorff approximations $\iota_n\colon(Y,y)\to(\lambda_n A,x)$. Then, for $a_n=\iota_n(y')$,

$$(\lambda_n A,a_n,\lambda_n(\phi-\phi(a_n)))\to(Y,y',\psi-\psi(y')). \tag{5.7}$$

However, each $a_n\in A$ and so there exists $(Y_n,y_n,\psi_n)\in\operatorname{Tan}(X,a_n,\phi)$ and a $y_n'\in Y_n$ such that $(Y_n,y_n',\psi_n-\psi_n(y_n'))\in B$ and

$$d((Y_n,y_n',\psi_n-\psi_n(y')),(\lambda_n A,a_n,\lambda_n(\phi-\phi(a_n))))>\delta \tag{5.8}$$

whenever $\lambda_n>1/\delta$. In particular, since B has diameter $\delta/4$,

$$d((Y_n,y_n',\psi_n-\psi_n(y_n')),(Y,y',\psi-\psi(y')))<\delta/4 \tag{5.9}$$

for each $n\in\mathbb{N}$. Therefore, for any $\lambda_n>1/\delta$,

$$\begin{aligned}\delta &\overset{(5.8)}{<} d((Y_n,y_n',\psi_n-\psi_n(y')),(\lambda_n A,a_n,\phi-\phi(a_n)))\\ &\le d((Y_n,y_n',\psi_n-\psi_n(y_n'))),(Y,y',\psi-\psi(y')))\\ &\quad + d((Y,y',\psi-\psi(y')),(\lambda_n A,a_n,\lambda_n(\phi-\phi(a_n))))\\ &\overset{(5.9)}{\le} \delta/4+d((Y,y',\psi-\psi(y')),(\lambda_n A,a_n,\lambda_n(\phi-\phi(a_n)))) \overset{(5.7)}{\to} \delta/4\end{aligned}$$

a contradiction. □

Finally, we apply the theory of Gromov–Hausdorff tangents to the setting of metric measure spaces with Alberti representations. For X a metric space and $x \in X$, a *line* passing through x is an isometry $g\colon \mathbb{R} \to X$ that contains x. For $v \in \mathbb{R}^n$, such a line is *in the direction of* v (with respect to some $\phi\colon X \to \mathbb{R}^n$) if $\phi(g(\mathbb{R})) = \mathbb{R}v$.

Lemma 5.7. *Let (X, d, μ) be a metric measure space, $\phi\colon X \to \mathbb{R}^n$ Lipschitz and $C \subset \mathbb{R}^n$ a cone. Suppose that, for some $A \subset X$, $\mu|_A$ has a C-Alberti representation with respect to ϕ. Then for μ almost every $x \in A$ there exists a $v \in C \setminus \{0\}$ such that, for every $(Y, y, \psi) \in \operatorname{Tan}(X, x, \phi)$, Y contains a line passing through y in the ψ direction of v.*

Proof. From the definition of an Alberti representation, we know for μ almost every $x \in A$, there exists a $\gamma \in \Gamma(X)$ and $t \in \operatorname{dom}\gamma$ such that $\gamma(t) = x$, t is a density point of $\operatorname{dom}\gamma$ and $(\phi \circ \gamma)'(t) \in C \setminus \{0\}$. Fix such a point $x \in A$.

Now suppose that $(Y, y, \psi) \in \operatorname{Tan}(X, x, \phi)$ and that $\iota_n\colon (Y, y) \to (\lambda_n X, x)$ are Hausdorff approximations. Since $(\phi \circ \gamma)'(t)$ exists, by unravelling the definitions, it is straightforward (but a bit tedious) to show that there exists a map $g\colon \mathbb{R} \to Y$ containing y such that $(\psi \circ g)'(s) = (\phi \circ \gamma)'(t)$ for all $s \in \mathbb{R}$. By reparametrising g, we see that it is a line passing through y in the direction of $v := (\phi \circ \gamma)'(t) \in C \setminus \{0\}$. □

By considering many independent Alberti representations we obtain the following.

Corollary 5.8 ([Dav15], Corollary 5.2)**.** *Let (X, d, μ) be a doubling metric measure space and $\phi\colon X \to \mathbb{R}^n$ Lipschitz. Suppose that, for some $A \subset X$, $\mu|_A$ has n independent Alberti representations with respect to ϕ. Then for μ almost every $x \in A$ and every $(Y, y, \psi) \in \operatorname{Tan}(X, x, \phi)$, $\psi\colon Y \to \mathbb{R}^n$ is surjective.*

Proof. By applying Lemma 5.7 to each of the Alberti representations in the hypothesis, we know that the following is true. For μ almost every $x \in A$ there exist linearly independent $v_1, \ldots, v_n \in \mathbb{R}^n$ such that, for every $(Y, y, \psi) \in \operatorname{Tan}(X, x, \phi)$, Y contains lines $g_1, \ldots, g_n$ each passing through y in the ψ direction of $v_1, \ldots, v_n$ respectively.

Since (X, d, μ) is doubling, we can apply Theorem 5.6. This implies that, for μ almost every $x \in A$, the following is true. There exist linearly independent $v_1, \ldots, v_n \in \mathbb{R}^n$ such that, for every $(Y, y, \psi) \in \operatorname{Tan}(X, x, \phi)$ and any $y' \in Y$, Y contains lines $g_1, \ldots, g_n$ each passing through y' in the ψ direction of $v_1, \ldots, v_n$ respectively.

This implies that, for any $(Y, y) \in \operatorname{Tan}(X, x)$, $\psi(Y) = \mathbb{R}^n$. Indeed, since (Y, y) contains a line passing through y in the direction of v_1, $\psi(Y)$ contains $V_1 := \psi(y) + \mathbb{R}v_1$. Now, from any point $z \in V_1$, there exists $y' \in Y$ such that $\psi(y') = z$. Since Y contains a line passing through y' in the direction of v_2, $\psi(Y)$ contains $\psi(y') + \mathbb{R}v_2$ and hence $V_2 := \psi(y) + \mathbb{R}v_1 + \mathbb{R}v_2$. Repeating iteratively, we see that $\psi(Y)$ contains

$$V_n := \psi(y) + \mathbb{R}v_1 + \mathbb{R}v_2 + \ldots + \mathbb{R}v_n = \mathbb{R}^n$$

as required. □

This gives us the required bound on the number of independent Alberti representations by the doubling constant.

Theorem 5.9. *Let (X, d, μ) be a doubling metric measure space. There exists a $N \in \mathbb{N}$ depending only on $\operatorname{doub}(\mu)$ such that, for any $A \subset X$ with $\mu(A) > 0$, $\mu|_A$ can have at most N independent Alberti representations.*

Proof. Suppose that $(Y, y) \in \mathrm{Tan}(X, x)$. Since Y is doubling with $\mathrm{doub}(Y) \leq \mathrm{doub}(X)$, $\dim_H Y \leq \log_2 \mathrm{doub}(X)$ by Observation 2.5. In particular, a Lipschitz function cannot map Y onto $\mathbb{R}^n$ with $n > \log_2 \mathrm{doub}(X)$. Since (X, d, μ) is doubling, the conclusion follows from Corollary 5.8 and Lemma 2.4. □

To conclude we summarise the proof of Cheeger's theorem.

Proof of Theorem 3.3. Let (X, d, μ) be a PI space. By Proposition 3.4 it suffices to find an $N \in \mathbb{N}$ for which, whenever $\phi\colon X \to \mathbb{R}^n$ is Lipschitz satisfying eq. (3.1) at all points $x \in U$ with $\mu(U) > 0$, we must have $n \leq N$. Since X satisfies a Poincaré inequality, by applying Theorem 4.8, there exists a countable decomposition $U = \cup_i U_i$ such that each $\mu|_{U_i}$ has n independent Alberti representations. Thus, if we find a bound N to the number of independent of Alberti representations of any positive measure subset of X, we are done. This is given by Theorem 5.9. □

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