AN INTRODUCTION TO RECTIFIABILITY IN METRIC SPACES

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Throughout these notes, (X, d) will denote a metric space. By a *measure* on X we mean a countably sub-additive set function μ defined on the power set of X with $\mu(\emptyset) = 0$. If μ is a measure on X then $A \subset X$ is *μ-measurable* if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$$

for all $E \subset X$. Any $A \subset X$ with $\mu(A) = 0$ is μ-measurable and the μ-measurable subsets of X form a σ-algebra. We will assume that all measures are *Borel measures*, meaning that all Borel sets are measurable.

1. Rectifiable subsets of a metric space

The prospect of studying the concepts of geometric measure theory in an arbitrary metric space stems from the fact that the fundamental definitions, namely Hausdorff measure and Lipschitz functions, do not rely on Euclidean structure.

For $s, \delta \geq 0$ and $E \subset X$, we write

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i\in\mathbb{N}} \operatorname{diam}(S_i)^s : E \subset \bigcup_{i\in\mathbb{N}} S_i,\ \operatorname{diam} S_i \leq \delta \right\}$$

and define the *s-dimensional Hausdorff measure* of E by

$$\mathcal{H}^s(E) := \sup_{\delta\to 0} \mathcal{H}^s_\delta(E).$$

Note that, since $\mathcal{H}^s_\delta(E)$ decreases as δ decreases, the supremum is in fact a limit as $\delta \to 0$.

Recall that a map $f\colon (X, d) \to (Y, \rho)$ between metric spaces is *L-Lipschitz*, for $L \geq 0$, if

$$\rho(f(x), f(y)) \leq L d(x, y)$$

for all $x, y \in X$. The least such L is called the *Lipschitz constant* of f and will be denoted by $\operatorname{Lip}(f)$. An injective Lipschitz function is *bi-Lipschitz* if its inverse is also Lipschitz; it's *bi-Lipschitz* constant is $\operatorname{Lip}(f) \cdot \operatorname{Lip}(f^{-1})$.

Definition 1.1. A $\mathcal{H}^s$-measurable $E \subset X$ is *n-rectifiable* if, for each $i \in \mathbb{N}$, there exist $A_i \subset \mathbb{R}^n$ and a Lipschitz $f_i\colon A_i \to X$ such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i\in\mathbb{N}} f_i(A_i)\right) = 0.$$

Note that we do not ask, as is often done in classical geometric measure theory, that each $A_i = \mathbb{R}^n$. This is to avoid topological obstructions in X. If $X = \mathbb{R}^m$ equipped with a norm, then by the McShane extension theorem (see Exercise 1.1), an equivalent definition is obtained if we require each $A_i = \mathbb{R}^n$. More generally, if X is a Banach space, the same conclusion is obtained using a Lipschitz extension result of Johnson, Lindenstrauss, and Schechtman [14]. If X is complete, an equivalent definition of rectifiability is obtained if we require the A_i to be compact (see Exercise 1.2).

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In classical geometric measure theory, geometric descriptions of rectifiable sets fundamentally rely on Rademacher's theorem.

Theorem 1.2 (Lebesgue $n = 1$, Rademacher $n \geq 2$)**.** *A Lipschitz $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable $\mathcal{L}^n$-a.e.*

Analogously, we will see that the geometry of rectifiable subsets of a metric space rely on a "metric Rademacher" theorem.

To begin we consider the case that our Lipschitz function is defined on the whole of $\mathbb{R}^n$.

1.1. **Rectifiable curves.**

Definition 1.3. Let $\gamma\colon [a,b] \to X$ be a Lipschitz curve. The *variation* of γ is defined as

$$\mathrm{Var}(\gamma) := \left\{ \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \right\},$$

where the supremum ranges over all

$$a \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq b.$$

Further, we define the *metric speed* of γ at t as

$$|\dot{\gamma}| := \lim_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|}$$

whenever the limit exists.

We begin with the proof of the metric Rademacher theorem in the case $n = 1$, and follow the proof from [3, Theorem 4.1.6].

Proposition 1.4 (Ambrosio [1], Kirchheim [16])**.** *Let $\gamma\colon [a,b] \to X$ be Lipschitz. For $\mathcal{L}^1$-a.e. $t \in [a,b]$, $|\dot{\gamma}|(t)$ exists and*

$$\mathrm{Var}(\gamma) = \int_a^b |\dot{\gamma}| \, \mathrm{d}\mathcal{L}^1. \tag{1.1}$$

Proof. Let x_n, $n \in \mathbb{N}$, be a dense subset of $\gamma([a,b])$ and , for each $n \in \mathbb{N}$, let

$$\phi_n(t) = d(\gamma(t), x_n),$$

a 1-Lipschitz function $\phi_n\colon [a,b] \to \mathbb{R}$. Thus, by Theorem 1.2, $\phi_n'(t)$ exists, for each $n \in \mathbb{N}$ and $\mathcal{L}^1$-a.e. $t \in [a,b]$. We set

$$m(t) = \sup_{n \in \mathbb{N}} |\phi_n'(t)|$$

and will show that $|\dot{\gamma}|(t) = m(t)$ for $\mathcal{L}^1$-a.e. $t \in [a,b]$.

Since each $x \mapsto d(x, x_n)$ is 1-Lipschitz, we have

$$\begin{aligned} \liminf_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|} &\geq \liminf_{s \to t} \frac{|\phi_n(s) - \phi_n(t)|}{|s-t|} \\ &= |\phi_n'(t)| \end{aligned}$$

for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Taking the supremum over $n \in \mathbb{N}$ gives

$$\liminf_{s \to t} \frac{d(\gamma(s), \gamma(t))}{|s-t|} \geq m(t) \tag{1.2}$$

for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

On the other hand, since the x_n are dense, for any $s, t \in \mathbb{R}$,

$$d(\gamma(s), \gamma(t)) = \sup_{n \in \mathbb{N}} |d(\gamma(s), x_n) - d(x_n, \gamma(t))| = \sup_{n \in \mathbb{N}} |\phi_n(s) - \phi_n(t)|.$$

Further, since each ϕ_n is a Lip(γ)-Lipschitz real valued function on $\mathbb{R}$,

$$\begin{aligned} d(\gamma(s),\gamma(t)) &\leq \sup_{n\in\mathbb{N}} \int_s^t |\phi_n'(\tau)|\,\mathrm{d}\tau \\ &\leq \int_s^t m(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{1.3}$$

Finally, $|\phi_n'(t)| \leq \operatorname{Lip}(\gamma)$ whenever it exists. Therefore $|m(t)| \leq \operatorname{Lip}(\gamma)$ too and hence is locally integrable. If t is a Lebesgue point of m then

$$\limsup_{s\to t} \frac{d(\gamma(s),\gamma(t))}{|s-t|} \leq m(t).$$

Combining this inequality with (1.2) completes the first part of the proof.

In order to prove (1.1), note that (1.3) gives

$$\sum_{i=1}^{n-1} d(\gamma(t_{i+1}),\gamma(t_i)) \leq \int_a^b |\dot\gamma|\,\mathrm{d}\mathcal{L}^1$$

for any $a \leq t_1,\ldots,t_n \leq b$. Taking the supremum over such partitions gives one of the required inequalities for (1.1). For the opposite inequality, let $\epsilon > 0$, $h = (b-a)/n$ and $t_i = a + ih$, for a choice of $n \in \mathbb{N}$ for which $h \leq \epsilon$. Then

$$\begin{aligned} \frac{1}{h}\int_a^{b-\epsilon} d(\gamma(t+h),\gamma(t))\,\mathrm{d}t &\leq \frac{1}{h}\int_0^h \sum_{i=0}^{n-2} d(\gamma(\tau+t_{i+1}),\gamma(\tau+t_i))\,\mathrm{d}\tau \\ &\leq \frac{1}{h}\int_0^h \operatorname{Var}(\gamma)\,\mathrm{d}\tau = \operatorname{Var}(\gamma). \end{aligned}$$

Thus, Fatou's lemma and the arbitrariness of $\epsilon > 0$ gives the other inequality. □

As in the classical case, the above proposition allows us to find arc length parametrisations of rectifiable curves.

Corollary 1.5. *Let $\gamma\colon [0,1] \to X$ be Lipschitz and set $l = \operatorname{Var}(\gamma)$. There exists an increasing $\phi\colon [0,l] \to [0,1]$ such that $\tilde\gamma := \gamma\circ\phi$ is 1-Lipschitz and satisfies $|\dot{\tilde\gamma}|(t) = 1$ for $\mathcal{L}^1$-a.e. $t \in [0,l]$.*

Proof. For each $t \in [0,1]$ let

$$\psi(t) = \int_0^t |\dot\gamma|\,\mathrm{d}\mathcal{L}^1 = \operatorname{Var}(\gamma|_{[0,t]})$$

which is Lipschitz, increasing and satisfies $\psi(0) = 0$ and $\psi(1) = l$. Note that $\dot\psi = |\dot\gamma|$ almost everywhere. For $s \in [0,l]$ define

$$\phi(s) = \inf\{t \in [0,1] : \psi(t) = s\},$$

which is also increasing and satisfies $\psi\circ\phi = \mathrm{id}$. If $\tilde\gamma = \gamma\circ\phi$ then (1.1) gives

$$\begin{aligned} d(\tilde\gamma(s),\tilde\gamma(t)) &= d(\gamma(\phi(s)),\gamma(\phi(t))) \\ &\leq \int_{\phi(s)}^{\phi(t)} |\dot\gamma|\,\mathrm{d}\mathcal{L}^1 \\ &= \int_{\phi(s)}^{\phi(t)} \dot\psi\,\mathrm{d}\mathcal{L}^1 \\ &= \int_{\psi(\phi(s))}^{\psi(\phi(t))} \operatorname{card}\psi^{-1}\,\mathrm{d}\mathcal{L}^1 \\ &= \int_s^t \operatorname{card}\psi^{-1}\,\mathrm{d}\mathcal{L}^1 = t-s, \end{aligned}$$

where the second equality follows from the (classical) area formula, and the final equality holds since the set of x with card $\psi^{-1}(x) > 1$ is at most countable (since $\psi^{-1}(x)$ contains an open interval). Thus $\tilde{\gamma}$ is 1-Lipschitz.

Finally, let N be the null set where at least one of ψ or γ is not (metrically) differentiable and let $M = \phi^{-1}(N)$. Since $\psi \circ \phi = \text{id}$, $M \subset \psi(N)$ and hence, since ψ is Lipschitz, M is a null set. Moreover, by the chain rule, for any $t \notin M$,

$$|\dot{\tilde{\gamma}}(t)| = |\dot{\gamma}|(\phi(t))|\dot{\phi}(t)| = |\dot{\gamma}|(\phi(t))|(\dot{\psi^{-1}})(\phi(t))| = 1.$$

□

1.2. **Higher dimensional rectifiable sets.** The content of the remainder of this section is due to Kirchheim [16].

Definition 1.6. Let $f\colon \mathbb{R}^n \to X$. A *metric derivative* of f at x is a seminorm $|Df|(x)$ on $\mathbb{R}^n$ such that

$$\lim_{y,z\to x} \frac{|d(f(y),f(z)) - |Df(x)|(y-z)|}{\|y-x\| + \|z-x\|} = 0. \tag{1.4}$$

Note that, if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x, then $|Df|(x) = \|Df(x)\|$.

Theorem 1.7. *Let $f\colon \mathbb{R}^n \to X$ be Lipschitz. For $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, $|Df|(x)$ exists.*

Proof. Let u_m, $m \in \mathbb{N}$, be a dense subset of $\mathbb{S}^{n-1}$. By Proposition 1.4, for almost every $x \in \mathbb{R}^n$,

$$|Df|(x)(u_m) := \lim_{t\to 0} \frac{d(f(x+tu_m), f(x))}{|t|} \tag{1.5}$$

exists and is finite. Moreover, for such an x, $|Df|(x)$ is Lip(f)-Lipschitz on

$$\{u_j : j \in \mathbb{N}\}.$$

Indeed, for any $i,j \in \mathbb{N}$ and $\epsilon > 0$,

$$\begin{aligned} ||Df|(x)(u_i) - |Df|(x)(u_j)| &\le \left| \frac{d(f(x+tu_i),f(x))}{|t|} - \frac{d(f(x+tu_j),f(x))}{|t|} \right| + 2\epsilon \\ &\le \frac{d(f(x+tu_i), f(x+tu_j))}{|t|} + 2\epsilon \\ &\le \text{Lip}(f)\frac{\|tu_i - tu_j\|}{|t|} + 2\epsilon \end{aligned}$$

provided t is sufficiently small. Hence $|Df|(x)$ may be extended to the whole of $\mathbb{S}^{n-1}$ (by Exercise 1.2). Further, we extend $|Df|(x)$ to all of $\mathbb{R}^n$ by defining

$$|Df|(x)(\lambda u) = |\lambda||Df|(x)(u) \tag{1.6}$$

for any $\lambda \in \mathbb{R}$ and $u \in \mathbb{S}^{n-1}$. In particular, by (1.5),

$$|Df|(x)(u) = \lim_{t\to 0} \frac{d(f(x+tu),f(x))}{|t|}$$

for all $u \in \mathbb{R}^n$.

Now observe that, by the Lebesgue density and Lusin theorems, it suffices to prove the result for x a density point of a compact set K on which $|Df|$ is continuous. For such an x we show that

$$\lim_{t\to 0} \frac{d(f(x+tu), f(x+tu'))}{|t|} = |Df|(x)(u-u') \tag{1.7}$$

uniformly for $u, u' \in B(0,1)$. Fix $\epsilon > 0$ and $u, u' \in B(0,1)$. Since $|Df|$ is (uniformly) continuous on $K \times B(0,1)$, there exists $r > 0$ such that, for any $y, z \in B(x,r) \cap K$,

$$||Df|(y)(u) - |Df|(z)(u)| \le \epsilon$$

and

$$\left| \frac{d(f(y+tu), f(y))}{|t|} - |Df|(y)(u) \right| \leq \epsilon$$

for all $u \in B(0,1)$ and $|t| < r$. Further, since x is a density point of K, provided t is sufficiently small, there exists $y \in K \cap B(x+tu', \epsilon t)$. Thus

$$\begin{aligned}
&\left| \frac{d(f(x+tu), f(x+tu'))}{|t|} - |Df|(x)(u-u') \right| \\
&\quad \leq \left| \frac{d(f(y+t(u-u')), f(y))}{|t|} - |Df|(y)(u-u') \right| + 2\operatorname{Lip}(f)\epsilon + \epsilon \\
&\quad \leq 2\epsilon(1+\operatorname{Lip}(f)),
\end{aligned}$$

which gives (1.7). Note that (1.7) implies (1.4) and the fact that $|Df|(x)$ satisfies the triangle inequality. Combining this with (1.6) shows that $|Df|(x)$ is a seminorm. □

1.3. **Lipschitz functions on arbitrary domains.** Next we extend Theorem 1.7 to the case when f is only defined on a subset of $\mathbb{R}^n$. To do this we use the Kuratowski embedding.

Lemma 1.8 (Kuratowski embedding)**.** *Any separable metric space isometrically embeds into ℓ_∞, the set of bounded sequences equipped with the supremum norm.*

For the proof, see Exercise 1.3.

Corollary 1.9. *Let $S \subset \mathbb{R}^n$ and $f\colon S \to X$ be Lipschitz. Then f is differentiable $\mathcal{L}^n$-a.e. in S. More precisely, for $\mathcal{L}^n$-a.e. $x \in S$, there exists a unique norm $|Df|(x)$ on $\mathbb{R}^n$ such that* (1.4) *holds for $y, z \in S$.*

Proof. Since S is separable and f is Lipschitz, $f(S)$ is separable. We identify $f(S)$ with its isometric image in ℓ_∞ given by Lemma 1.8. For each $n \in \mathbb{N}$, the function $f_n\colon S \to \mathbb{R}$ defined by $f_n(s) = f(s)_n$ is $\operatorname{Lip}(f)$-Lipschitz. Using the McShane extension theorem (Exercise 1.1), we extend each f_n to a $\operatorname{Lip}(f)$-Lipschitz function $\tilde{f}_n\colon \mathbb{R}^n \to \mathbb{R}$ and define $\tilde{f}\colon \mathbb{R}^n \to \ell_\infty$ by $(\tilde{f})_n = f_n$ for each $n \in \mathbb{N}$. Then $\tilde{f}$ is $\operatorname{Lip}(f)$-Lipschitz and agrees with f on S. Therefore, if $\tilde{f}$ is differentiable at x, (1.4) holds for f and $y, z \in S$. It remains to show uniqueness of the derivative, which holds at any density point of S, see Exercise 1.4. □

Definition 1.10. Let $f\colon S \subset \mathbb{R}^n \to X$ be Lipschitz. We say that $x \in S$ is a *regular* point of f if x is a density point of S, $|Df|(x)$ exists and it is a norm. Standard arguments show that the set of regular points of a Borel set S is a Borel subset of S.

1.4. **Properties of rectifiable subsets of a metric space.**

Lemma 1.11. *Let $S \subset \mathbb{R}^n$ be Borel, $f\colon S \to X$ Lipschitz and $\lambda > 1$. There exist Borel sets $E_i \subset S$ and norms $\|\cdot\|_i$ on $\mathbb{R}^n$ such that*

- *The set of regular points of f equals $\cup_i E_i$;*
- *For each $i \in \mathbb{N}$, $f\colon (E_i, \|\cdot\|_i) \to X$ is λ-bi-Lipschitz.*

Proof. First note that the space of all norms on $\mathbb{R}^n$ is separable, in the following sense. There exists a countable set of norms $\mathcal{N}$ such that, for any $\lambda > 0$ and any norm $\|\cdot\|$ on $\mathbb{R}^n$, there exists $\|\cdot\|' \in \mathcal{N}$ such that

$$\frac{1}{\lambda}\|v\|' \leq \|v\| \leq \lambda\|v\|' \quad \forall v \in \mathbb{R}^n. \tag{1.8}$$

This follows, for example, from the separability of all convex, compact symmetric subset of $\mathbb{R}^n$ equipped with the Hausdorff metric. Enumerate $\mathcal{N}$ as $\|\cdot\|_1, \|\cdot\|_2, \ldots$.

Now define E_1 to be those regular points of f for which (1.8) is satisfied with $\|\cdot\|' = \|\cdot\|_1$ and $\|\cdot\| = |Df|(x)$, a Borel set. Define E_2 to be those regular points of f not in E_1 for which (1.8) is satisfied with $\|\cdot\|' = \|\cdot\|_2$ and $\|\cdot\| = |Df|(x)$, a Borel set. Repeating inductively we obtain a Borel decomposition of the regular points of f.

Now fix $i \in \mathbb{N}$ and $x \in E_i$. If $y \in S$ is sufficiently close to x then

$$\begin{aligned}\frac{1}{\lambda^2}\|y-x\|_i \leq \frac{1}{\lambda}Df(x)(y-x) &\leq d(f(y),f(x)) \\ &\leq \lambda Df(x)(y-x) \leq \lambda^2\|y-x\|_i. \end{aligned} \tag{1.9}$$

The map

$$R(x) := \sup\{r : \text{extreme inequalities in (1.9) hold } \forall y \in U(x,r)\cap S\}$$

is upper semi-continuous. Thus, we may decompose E_i into countably many Borel sets E_i^j on which $R(x) \geq 1/j$. By further decomposing each E_j^j into sets of diameter at most $1/j$, we see that f is λ^2-bi-Lipschitz on each set of the decomposition. □

The set of irregular (non-regular) points of a Lipschitz function are handled with the following Sard type result.

Lemma 1.12. *Let $S \subset \mathbb{R}^n$ be Borel and $f\colon S \to X$ Lipschitz. If $I \subset S$ is the set of irregular points of f, then $\mathcal{H}^n(f(I)) = 0$.*

Proof. The set of points where f is not differentiable has measure zero, as is the set of non-density points of S. The image of these points under f has Hausdorff measure zero (see Exercise 1.5). Therefore it suffices to prove that the set I of points where $|Df|(x)$ exists but is not a norm satisfies $\mathcal{H}^n(f(I)) = 0$. Second, it suffices to prove the result for S bounded, say $S \subset B(0,1)$.

Fix $\epsilon > 0$. If $x \in I$ then there exists $v_x \in \mathbb{S}^{n-1}$ with $|Df|(x)(v_x) = 0$. That is, there exists $r_x > 0$ such that

$$|d(f(y),f(z)) - |Df|(x)((y-z)\cdot v_x^\perp)| \leq \epsilon(\|y-x\| + \|z-x\|) \tag{1.10}$$

for all $y,z \in B(x,r_x)\cap S$. In particular, $f(B(x,r)\cap I)$ is contained within the ϵr neighbourhood of

$$f(B(x,r)\cap I \cap (x+v_x^\perp))$$

for all $0<r<r_x$. Therefore, $f(I\cap B(x,r))$ is contained within $[\epsilon^{-(n-1)}]$ many balls of radius ϵr. In particular,

$$\mathcal{H}^n_{\mathrm{Lip}(f)\epsilon r}(f(I\cap B(x,r))) \leq [\epsilon^{-(n-1)}](2\epsilon r)^n \leq 4^n \epsilon r^n. \tag{1.11}$$

Fix $\delta > 0$. Consider the collection of balls $\mathcal{B}$ of the form $B(x,r)$ with $x \in I$ and $r < \min\{r_x, \delta/\mathrm{Lip}(f)\epsilon\}$. Then $\mathcal{B}$ is a Vitali cover of I. Applying the Vitali covering theorem[1] gives a disjoint sub-collection $B(x_1,r_1), B(x_2,r_2),\ldots$ with

$$\mathcal{L}^n\left(I \setminus \bigcup_{i\in\mathbb{N}} B(x_i,r_i)\right) = 0.$$

Applying (1.11) to each $B(x_i,r_i)$ gives

$$\mathcal{H}^n_\delta(f(I)) \leq \sum_{i\in\mathbb{N}} \mathcal{H}^n_\delta(f(I\cap B(x_i,r_i))) \leq 4^n \sum_{i\in\mathbb{N}} \epsilon r_i^n,$$

However, the $B(x_i,r_i)$ are disjoint subsets of $B(0,1+\delta)$ and so $\sum_i r_i^n \leq (1+\delta)^n$. Since $\epsilon > 0$ is arbitrary, this shows that $\mathcal{H}^n_\delta(f(I)) = 0$ and hence $\mathcal{H}^n(f(I)) = 0$. □

Combining Lemmas 1.11 and 1.12 gives the following.

[1] We will prove the Vitali covering theorem for doubling measures in Theorem 3.9.

Corollary 1.13. *Let X be complete and $E \subset X$ n-rectifiable. For any $\epsilon > 0$ there exist norms $\|\cdot\|_i$ on $\mathbb{R}^n$, Borel sets $A_i \subset \mathbb{R}^n$ and $(1+\epsilon)$-bi-Lipschitz maps $f_i \colon (A_i, \|\cdot\|_i) \to X$ such that*

$$\mathcal{H}^n\left(E \setminus \bigcup_{i\in\mathbb{N}} f_i(A_i)\right) = 0.$$

Remark 1.14. From Corollary 1.13 one can deduce area formulae for rectifiable metric spaces, see [16, Theorem 7] and [2, Theorem 5.1, Theorem 8.2].

We conclude this section with Kirchheim's theorem on the local structure of rectifiable metric spaces.

Theorem 1.15. *Let X be complete and $E \subset X$ be n-rectifiable. For $\mathcal{H}^n$-a.e. $x \in E$ there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^n$, a map $\phi_x \colon E \to \mathbb{R}^n$ and a compact $A_x \subset E$ such that*

$$\lim_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap A_x)}{(2r)^n} = 1 \tag{1.12}$$

and

$$\lim_{r\to 0}\sup\left\{\left|1 - \frac{\|\phi_x(y)-\phi_x(z)\|_x}{d(y,z)}\right| : y \neq z \in A_x \cap B(x,r)\right\} = 0. \tag{1.13}$$

Proof. It suffices to prove the result for $E = f(S)$, for $S \subset \mathbb{R}^n$ Borel and $f \colon S \to X$ Lipschitz. For each $k \in \mathbb{N}$ let $S = \cup_{i\in\mathbb{N}} E_i^k$ and norms $\|\cdot\|_i^k$ be obtained from Lemma 1.11 for some decreasing $\lambda_k > 1$ with $\lambda_k \to 1$. Let G_i^k be the set of Lebesgue density points of each E_i^k. If $S' = \cap_k \cup_i G_i^k$, we claim the conclusion holds for any $f(a) \in f(S')$, which suffices since $\mathcal{H}^n(S \setminus S') = 0$.

To this end, for each $k \in \mathbb{N}$ let $i(k) \in \mathbb{N}$ be such that $a \in G_i^k$. The conclusion of Lemma 1.11 implies that

$$\frac{1}{\lambda_k \lambda_{k+1}}\|\cdot\|_k \leq \|\cdot\|_{k'} \leq \lambda_k\lambda_{k+1}\|\cdot\|_k$$

for all $k' \geq k \geq 1$. Therefore, there exists a limiting norm $\|\cdot\|_x = \lim_{k\to\infty}\|\cdot\|_k$. Moreover,

$$\frac{1}{\lambda_k\lambda_{k+1}}\|y-z\|_x \leq d(f(y),f(z)) \leq \lambda_k\lambda_{k+1}\|y-z\|_x$$

for all $y, z \in E_i^{i(k)}$. Since a is a density point of each $E_i^{i(k)}$, there exist $r_k \to 0$ such that

$$\mathcal{H}^n(B(a,r)\cap E_i^{i(k)}) > (1-1/k)\mathcal{H}^n(B(a,r))$$

for all $0 \leq r < r_k$. We set $A_k = E_i^{i(k)} \cap B(a,r_k) \setminus B(a,r_{k+1})$. By reducing the measure of each A_k slightly, we may suppose that they are compact. Therefore, $A := \cup_k A_k \cup \{a\}$ is compact and

$$\frac{\mathcal{H}^n(B(a,r\cap A))}{\mathcal{H}^n(B(a,r))} \to 1 \quad \text{as } r \to 0 \tag{1.14}$$

by construction.

Finally, define $A_x = f(A)$, a compact set. Lemma 1.11 and (1.14) imply (1.12). Also define $\phi_x = f^{-1}$ on A_x, so that Lemma 1.11 implies (1.13). □

Remark 1.16. Theorem 1.15 allows us to define a *tangent norm* of E at almost every $x \in E$ as (an equivalence class in the Banach-Mazur compactum of) $\|\cdot\|_x$. See [16, Definition 10].

Corollary 1.17. *If $E \subset X$ is n-rectifiable with $\mathcal{H}^n(E) < \infty$ then*

$$\Theta^n(E,x) := \lim_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap E)}{(2r)^n} = 1$$

for $\mathcal{H}^n$-a.e. $x \in E$.

Proof. The fact that

$$\Theta^n_*(E,x) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap E)}{(2r)^n} \geq 1$$

is given by Theorem 1.15. The fact that

$$\Theta^{n,*}(E,x) := \limsup_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap E)}{(2r)^n} \leq 1 \tag{1.15}$$

for $\mathcal{H}^n(E) < \infty$ is a standard property of Hausdorff measure, see Lemma 3.11. □

1.5. **Exercises.**

Exercise 1.1. *Suppose that $\mathcal{F}$ is a collection of L-Lipschitz functions $f\colon X \to \mathbb{R}$.*

(1) *Show that $\sup \mathcal{F}\colon X \to \mathbb{R}$ defined by*

$$\sup \mathcal{F}(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

is also L-Lipschitz.

(2) *Let $A \subset X$ and $f\colon A \to \mathbb{R}$ be L-Lipschitz. Show that $\tilde{f} X \to \mathbb{R}$ defined by*

$$\tilde{f}(x) = \sup\{f(a) - Ld(x,a) : a \in A\}$$

is an L-Lipschitz extension of f.

(3) *Show that $\tilde{f}$ is the (pointwise) smallest L-Lipschitz extension of f to X.*

(4) *What can be said about $\hat{f}\colon X \to \mathbb{R}$ defined by*

$$\hat{f}(x) = \inf\{f(a) + Ld(x,a) : a \in A\}?$$

Exercise 1.2. *Let X, Y me metric spaces, $A \subset X$ and $f\colon A \to Y$ Lipschitz. Suppose that Y is complete.*

(1) *For any $x \in \bar{A}$, the closure of A, show that there exists a unique $y_x \in Y$ such that, if $x_n \to x$, then $f(x_n) \to y$.*

(2) *Extend f to $\bar{A}$ by defining $f(x) = y_x$. Show that $f\colon \bar{A} \to \mathbb{R}$ is $\operatorname{Lip}(f)$-Lipschitz.*

Exercise 1.3. *Let X be a separable metric space and x_n a countable dense subset of X. For each $x \in X$ define the sequence $\iota(x) \in \ell_\infty$ by*

$$\iota(x)_n = d(x, x_n) - d(x_n, x_0).$$

(1) *For any $x \in X$, show that $\iota(x)$ is bounded by $d(x, x_0)$.*

(2) *Show that ι is 1-Lipschitz.*

(3) *By considering $x_{n_k} \to x' \in X$, show that ι is an isometry.*

Exercise 1.4. (1) *Let $f\colon S \subset \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and x a density point of S. Show that there can be at most one $Df(x) \in L(\mathbb{R}^n, \mathbb{R})$ such that*

$$\lim_{S \ni y \to x} \frac{|f(y) - f(x) - Df(x)(y-x)|}{\|y-x\|} = 0.$$

Hint: *being linear, $Df(x)$ is determined by the behaviour of f near to the coordinate axes. More precisely, if $y \to x$ with*

$$\frac{\|\pi_{e_i^\perp}(y-x)\|}{\|y-x\|} \to 0,$$

show that

$$Df(x)(e_i) = \lim_{y \to x} \frac{f(y) - f(x)}{\|y - x\|}.$$

(2) *The situation is less simple for norms. Give an example of two (distinct) norms that agree on the coordinate axes.*

(3) *Let $f\colon S \subset \mathbb{R}^n \to X$ be Lipschitz and $x \in S$. Suppose $v \in \mathbb{S}^{n-1}$ and $y \to x$ with*

$$\frac{\|\pi_{v^\perp}(y - x)\|}{\|y - x\|} \to 0.$$

Show that, if it exists,

$$|Df|(x)(v) = \lim_{y \to x} \frac{d(f(y), f(x))}{\|y - x\|}.$$

(4) *Prove that the metric derivative of f (if it exists) is unique at any density point of S.*

Note that $\mathcal{L}^n$-a.e. $x \in S$ is a density point of S even if S is not measurable, see Exercise 3.8.

Exercise 1.5. *Let $f\colon X \to Y$ be a Lipschitz function between two metric spaces. For $S \subset X$ and $s \geq 0$ show that*

$$\mathcal{H}^s(f(S)) \leq \mathrm{Lip}(f)^s \mathcal{H}^s(S).$$

2. Sufficient conditions for rectifiability: the 1-dimensional case

Classically, one uses the ambient structure of Euclidean space, in particular the characterisation of rectifiability in terms of Lipschitz graphs, in order to establish sufficient conditions for rectifiability in terms of cones, see [18, Section 15]. Notably, this is the starting point for the proof of the Besicovitch–Federer projection theorem. Since this structure is not available to us, we introduce sufficient conditions on a set E that rely, in some sense, on the topology of E.

We begin with a classical result of Eilenberg and Harrold [11] which covers the case $n = 1$.

Lemma 2.1. *If $E \subset X$ is connected and $0 < r < \mathrm{diam}(E)/2$ then*

$$\mathcal{H}^1(E \cap B(x, r)) \geq r \quad \forall x \in E.$$

Proof. Pick $x \in E$ and let $\phi(y) = d(y, x)$ for all $y \in E$. Then ϕ is 1-Lipschitz and so

$$\mathcal{H}^1(\phi(E \cap B(x, r))) \leq \mathcal{H}^1(E \cap B(x, r)).$$

On the other hand, $\phi(E \cap B(x, r))$ contains an interval of length r (see Exercise 2.1), giving the required lower bound. □

Definition 2.2. Let $E \subset X$, $x, y \in E$ and $\epsilon > 0$. We say that x, y are *ϵ-connected* in E if there exist $x_1, \ldots, x_n \in E$ with $x_1 = x$ and $x_n = y$ such that $d(x_i, x_{i+1}) \leq \epsilon$ for each $1 \leq i < n$. In this case, $\{x_1, \ldots, x_n\}$ is called an *ϵ-chain* joining x to y in E.

Proposition 2.3. *Let $C \subset X$ be a compact and connected metric space with $\mathcal{H}^1(C) < \infty$. Then C is connected by injective rectifiable curves.*

Proof. Note that the statement of the proposition is invariant under isometries. Therefore, by Lemma 1.8, we may suppose that $C \subset \ell_\infty$. This gives us linear structure that we will use to construct Lipschitz curves.

For $\epsilon > 0$ and $x \in C$ let

$$C' = \{y \in C : x, y \text{ are } \epsilon\text{-connected in } C\}.$$

Then C' is both open and closed in C (see Exercise 2.2) and hence $C' = C$. That is, for every $x, y \in C$, there exists an ϵ-chain $\{x_0, \dots, x_n\}$ joining x to y. Moreover, by removing points from the chain if necessary, we may suppose that

$$d(x_i, x_j) \geq \epsilon \quad \text{whenever } |i - j| > 1.$$

This implies that distinct balls $B(x_i, \epsilon/2)$ and $B(x_j, \epsilon/2)$ are a positive distance apart whenever i, j are both even or both odd. Therefore

$$\mathcal{H}^1(C) \geq \sum_{i \text{ odd}} \mathcal{H}^1(C \cap B(x_i, \epsilon/2)).$$

and hence, if $\epsilon < \operatorname{diam}(C)$, Lemma 2.1 gives

$$\mathcal{H}^1(C) \geq \sum_{i \text{ odd}} \frac{\epsilon}{2}.$$

Combining this with the analogous inequality for even indices gives

$$2\mathcal{H}^1(C) \geq n\frac{\epsilon}{2}.$$

Now, because $C \subset \ell_\infty$, we can construct a 1-Lipschitz curve

$$\gamma_\epsilon \colon [0, 4\mathcal{H}^1(C)] \to X$$

connecting x to y by joining each x_i to x_{i+1} using line segments. In particular, γ_ϵ lies in the ϵ-neighbourhood of C. Since C is compact and the γ_ϵ are equicontinuous, Arzelà-Ascoli gives a Lipschitz curve γ joining x to y in C. By shortening γ if necessary, it may be taken to be injective, see Exercise 2.4 □

Theorem 2.4. *Let C be a compact, connected metric space with $\mathcal{H}^1(C) < \infty$. There exists a surjective 1-Lipschitz curve $\gamma \colon [0, 2\mathcal{H}^1(C)] \to C$.*

Proof. Throughout the proof we will make extensive use of Proposition 2.3 without explicit reference.

Since C is compact, there exist $x_0, y_0 \in C$ with $d(x, y) = \operatorname{diam} C =: d_0$. Let $\gamma_0 \subset C$ be an injective, Lipschitz curve connecting x to y. We inductively construct a sequence of curves $\gamma_n \subset C$ such that

- For each $i < n$, $\gamma_n \cap \gamma_i$ is a single point x_n;
- The other end point y_n of γ_n satisfies

$$d_n := d\left(y_n, \bigcup_{i<n} \gamma_i\right) = \max_{x \in C} d\left(x, \bigcup_{i<n} \gamma_i\right).$$

If $d_n = 0$ for some n then we have finitely many curves and stop. Otherwise, by Lemma 2.1, $\mathcal{H}^1(\gamma_n) \geq d_n$ and so

$$\sum_{i \geq 0} d_i \leq \sum_{i \geq 0} \mathcal{H}^1(\gamma_i) \leq \mathcal{H}^1(C).$$

For each $i \in \mathbb{N}$ set $l_i = \mathcal{H}^1(\gamma_i)$. By Exercise 2.4, we know that the arc length parametrisation of γ_i is defined on an interval of length l_i. We construct curves Γ_i as follows. On $[0, l_0]$ define Γ_0 to be the arc length parametrisation of γ_0 and on $[l_0, 2l_0]$ let Γ_0 equal the reverse parametrisation of γ_0. Thus Γ_0 is a loop, beginning at x_0, travelling to y_0 and returning to x_0. Next, we extend Γ_0 to a curve Γ_1 by adding a loop that begins at $x_1 \in \gamma$ travels along γ_1 to y_1, and then returns to x_1 by travelling along γ_1 in reverse. If we use the arc length parametrisation of γ_1, then Γ_1 is defined on $[0, 2(l_0 + l_1)]$. Inductively we construct Γ_n, a 1-Lipschitz closed loop on $[0, 2(\sum_{i \leq n} l_i)]$ whose image contains the image of each γ_i, $0 \leq i \leq n$. After extending each Γ_n to be constant on $(\sum_{i \leq n} l_i, \sum_{i \geq 0} l_i]$, we note that Γ_n is a Cauchy sequence in $C([0, \sum_{i \geq 0} l_i], C)$. Indeed, for any $t \in [0, \sum_{i \geq 0} d_i]$ and $n < m$,

$d(\Gamma_n(t), \Gamma_m(t)) \leq 2\sum_{i=n}^m l_i$. Thus Γ_n converges to some γ uniformly. In particular, γ is 1-Lipschitz.

Finally, by construction,

$$C \subset B(\Gamma_n, d_n) \subset B\left(\bigcup_{i\in\mathbb{N}} \Gamma_i, d_n\right),$$

for each $n \in \mathbb{N}$. Since $d_n \to 0$, C is contained in the closure of γ. Finally, since γ is compact, we have $C = \gamma$. □

Remark 2.5. Note that, except for countably many $x \in C$, $\gamma^{-1}(x)$ is at most 2 points. This is sharp, see Exercise 2.5.

2.1. **Exercises.**

Exercise 2.1. *Let $E \subset X$ be connected, $x \in E$ and $0 < r < \operatorname{diam}(E)/2$. For $\phi(y) = d(x,y)$, show that $\phi(E \cap B(x,r))$ contains an interval of length r.*

Exercise 2.2. *Show that C' in the proof of Proposition 2.3 is open and closed in C.*

Exercise 2.3. *Let C be connected with $\mathcal{H}^1(C) < \infty$. Show that C is totally bounded. Hint: For $0 < \epsilon < \operatorname{diam}(C)$ suppose we inductively choose*

$$x_n \in C \setminus \bigcup_{i=0}^{n-1} B(x_i, \epsilon).$$

whenever the set is non-empty. Use Lemma 2.1 to show that the process terminates after finitely many steps.

Therefore, in the hypotheses of Proposition 2.3, it would suffice to assume that C is a complete *and connected metric space with $\mathcal{H}^1(C) < \infty$.*

Exercise 2.4. *Let $\gamma_0 \colon [0, l_0] \to X$ be Lipschitz and define*

$$N = \{(s,t) \in [0,l_0]^2 : s < t,\ \gamma(s) = \gamma(t)\}.$$

1. *Show that $m = \max\{|s-t| : (s,t) \in N\}$ exists.*
2. *If γ_0 is not injective then $m > 0$. Let $s < t \in [0, l_0]$ with $|s-t| = m$ and define a Lipschitz curve $\gamma_1 \colon [0, l_0 - m] \to X$ by cutting out the loop defined by $\gamma|_{[s,t]}$. Note that γ_1 has the same end points as γ_0.*
3. *Assuming this process never terminates, inductively construct curves*

   $$\gamma_n \colon [0, l_n] \to X$$

   with the same end points as γ_0 and let $l = \lim_{n\to\infty} l_n$. Show that $\gamma_n|_{[0,l]}$ converges uniformly to some γ.
4. *Prove that γ is injective.*
5. *Suppose that $\gamma \colon [0,l] \to X$ is injective. Prove that*

   $$\mathcal{H}^1(\gamma([0,l])) = \int_0^l |\dot{\gamma}| \,\mathrm{d}\mathcal{L}^1.$$

Exercise 2.5. *For $n \in \mathbb{N}$ give an example of a compact connected metric space X_n with $\mathcal{H}^1(X_n) < \infty$ such that, for any γ as in the conclusion of Theorem 2.4, there exists x with* $\operatorname{card} \gamma^{-1}(x) = n$. *Give an example X for which infinitely many points have infinite preimage.*

3. Sufficient conditions for rectifiability: higher dimensions

David [10] gives conditions on an n-dimensional metric space X that ensures that a Lipschitz function $f\colon X \to \mathbb{R}^n$ may be decomposed into an explicit number of bi-Lipschitz pieces. This result was one of the corner stones of the development of *uniform* (or *quantitative*) rectifiability.

In this section we prove the following theorem, which shows that a condition similar to David's provides sufficient conditions for rectifiability. Note, however, that the proof of this qualitative case is *substantially* simpler than the quantitative results of David. To state our theorem, we require the following definitions.

Definition 3.1. For $s > 0$ and $A \subset X$, the upper and lower s-dimensional Hausdorff densities are defined as

$$\Theta^{*,s}(A,x) := \limsup_{r\to 0} \frac{\mathcal{H}^s((B(x,r)\cap A))}{(2r)^s}$$

and

$$\Theta^{s}_{*}(A,x) := \liminf_{r\to 0} \frac{\mathcal{H}^s((B(x,r)\cap A))}{(2r)^s}.$$

Theorem 3.2. *Let $f\colon X \to Y$ be Lipschitz, $\mathcal{H}^s(X) < \infty$ and, for $\mathcal{H}^s$-a.e. $x \in X$, suppose that*

$$\limsup_{r\to 0} \frac{\mathcal{H}^s(B(f(x),\lambda_x r)\setminus f(B(x,r)))}{(2\lambda_x r)^s} < \frac{1}{2}\Theta^s_*(Y,f(x)) \tag{3.1}$$

for some $0 < \lambda_x \leq 1$. Then there exists a countable Borel decomposition $X = N \cup \bigcup_i X_i$ with $\mathcal{H}^s(N) = 0$ such that each $f|_{X_i}$ is bi-Lipschitz.

First we demonstrate a key idea of David [10] that we will use in the proof of Theorem 3.2.

Definition 3.3. For $0 < \kappa < \lambda \leq 1$ and $0 < \xi < 1$, a function $f\colon V \subset X \to Y$ satisfies the condition $D(\lambda,\kappa,\xi)$ on a set $S \subset V$ if, for all $r < \operatorname{diam} S$,

$$\mathcal{H}^s(B(f(x),\lambda r)\setminus f(V\cap B(x,r))) < \frac{1}{2}\xi\mathcal{H}^s(B(f(x),\kappa r)).$$

Lemma 3.4. *Let $0 < \kappa < \lambda \leq 1$, $0 < \xi < 1$ and suppose $f\colon V \subset X \to Y$ satisfies $D(\lambda,\kappa,\xi)$ on $S \subset V$. Let $x,y \in S$, set $r = d(x,y)/4$ and suppose*

$$\mathcal{H}^s(B(f(x),\kappa r)) \geq \mathcal{H}^s(B(f(y),\kappa r)).$$

If

$$\mathcal{H}^s(f(V\cap B(x,r))\cap f(V\cap B(y,r))) \leq (1-\xi)\mathcal{H}^s\left(B\left(f(x),\kappa r\right)\right) \tag{3.2}$$

then

$$\rho(f(x),f(y)) \geq (\lambda-\kappa)\frac{d(x,y)}{4}. \tag{3.3}$$

Proof. Suppose that (3.3) does not hold. Then by the triangle inequality,

$$B\left(f(x),\lambda r\right)\cap B\left(f(y),\lambda r\right) \supset B\left(f(x),\kappa r\right).$$

Combining this with $D(\lambda,\kappa,\xi)$ negates (3.2). Indeed, it gives

$$\begin{aligned}\mathcal{H}^s(f(V\cap B(x,r))\cap f(V\cap B(y,r))) &> \mathcal{H}^s\left(B\left(f(x),\lambda r\right)\cap B\left(f(y),\lambda r\right)\right)\\ &\quad - \xi\mathcal{H}^s\left(B\left(f(x),\kappa r\right)\right)\\ &\geq (1-\xi)\mathcal{H}^s\left(B\left(f(x),\kappa r\right)\right).\end{aligned}$$

□

The remainder of the proof of Theorem 3.2 involves measure theoretic arguments to show that the domain of f may be decomposed into a countable number of sets satisfying $D(\lambda,\kappa,\xi)$.

We first prove some standard covering theorems and properties of the Hausdorff measure. We will use $B(x,r)$ to denote the closed ball in a metric space X centred at $x\in X$ with radius $r\geq 0$. Since the centre and radius of a ball are not uniquely defined by its elements, formally by a "ball" we mean a pair $(x,r)\in X\times(0,\infty)$, but in practice we mean the set of its elements. For a ball B and $\lambda>0$ we write λB for the ball with the same centre as B and λ times the radius.

Lemma 3.5 (Vitali covering lemma)**.** *Let X be a metric space and $\mathcal{B}$ an arbitrary collection of closed balls of uniformly bounded radii. There exists a disjoint sub-collection $\mathcal{B}'\subset\mathcal{B}$ such that any $B\in\mathcal{B}$ intersects a ball $B'\in\mathcal{B}'$ with*

$$\operatorname{rad} B'\geq \operatorname{rad} B/2.$$

In particular,

$$\bigcup_{B\in\mathcal{B}'} 5B\supset\bigcup_{B\in\mathcal{B}} B.$$

Proof. For each $n\in\mathbb{Z}$ let

$$\mathcal{B}_n=\{B\in\mathcal{B}: 2^n\leq \operatorname{rad} B<2^{n+1}\}.$$

Since the balls in $\mathcal{B}$ have uniformly bounded radii, the $\mathcal{B}_n$ are empty for all $n>N$, for some $N\in\mathbb{N}$. Let $\mathcal{B}'_N$ be a maximal disjoint sub-collection of $\mathcal{B}_N$. That is, the elements of $\mathcal{B}'_N$ are disjoint elements of $\mathcal{B}_N$ and if $B\in\mathcal{B}_N$, there exists a $B'\in\mathcal{B}'_N$ with $B\cap B'\neq\emptyset$. (In general such a maximal collection exists by Zorn's lemma. See also Exercise 3.2.) Let $\mathcal{B}'_{N-1}$ be a maximal collection such that $\mathcal{B}'_N\cup\mathcal{B}'_{N-1}$ is a disjoint collection. Repeat this for each $i\in\mathbb{N}$, obtaining a maximal collection $\mathcal{B}'_{N-i}$ such that $\mathcal{B}'_N\cup\ldots\cup\mathcal{B}'_{N-i}$ is a disjoint collection, and set $\mathcal{B}'=\bigcup_{n\leq N}\mathcal{B}'_n$.

Now suppose that $B\in\mathcal{B}$, say $B\in\mathcal{B}_n$. Then by construction there exists $B'\in\mathcal{B}'_m$ for some $m\geq n$ with $B\cap B'\neq\emptyset$. In particular, $\operatorname{rad} B'\geq\operatorname{rad} B/2$.

The final statement of the lemma follows from the triangle inequality. □

Definition 3.6. Let X be a metric space and $S\subset X$. A *Vitali cover* of S is a collection $\mathcal{B}$ of closed balls such that, for each $x\in S$ and each $\epsilon>0$, there exists a ball $B\in\mathcal{B}$ with $\operatorname{rad} B<\epsilon$ and $x\in B$.

Proposition 3.7. *Let X be a metric space, $S\subset X$ and suppose that $\mathcal{B}$ is a Vitali cover of S. Then there exists a disjoint $\mathcal{B}'\subset\mathcal{B}$ such that, for every finite $I\subset\mathcal{B}'$,*

$$S\setminus\bigcup_{B\in I} B\subset\bigcup_{B\in\mathcal{B}'\setminus I} 5B.$$

In particular, if $\mathcal{B}'=\{B_1,B_2,\ldots\}$ is countable (for example, if X is separable), then

$$S\setminus\bigcup_{i=1}^{n} B_i\subset\bigcup_{i>n} 5B_i$$

for each $n\in\mathbb{N}$.

Proof. Note that we may suppose $\mathcal{B}$ consists of balls with uniformly bounded radii. Let $\mathcal{B}'$ be a disjoint sub-collection of $\mathcal{B}$ obtained from Lemma 3.5. If $I\subset\mathcal{B}'$ is finite then

$$C:=\bigcup_{B\in I} B$$

is closed. Therefore, if $x \in S \setminus C$, since $\mathcal{B}$ is a Vitali cover of S, there exists $B \in \mathcal{B}$ with $x \in B$ such that $B \cap C = \emptyset$. However, B must intersect some $B' \in \mathcal{B}'$ with $\operatorname{rad} B' \geq \operatorname{rad} B/2$, and so $x \in 5B'$. That is, x belongs to

$$\bigcup_{B \in \mathcal{B}' \setminus I} 5B,$$

as required. □

Definition 3.8. A Borel measure μ on a metric space X is a *doubling measure* if there exists a $C_\mu \geq 1$ such that

$$0 < \mu(2B) \leq C_\mu \mu(B) < \infty$$

for all balls $B \subset X$.

Of course, we cannot mention covering theorems without proving the Vitali covering theorem.

Theorem 3.9 (Vitali covering theorem)**.** *Let μ be a doubling measure on a metric space X and let $\mathcal{B}$ be a Vitali cover of a set $S \subset X$. There exists a countable disjoint $\mathcal{B}' \subset \mathcal{B}$ such that*

$$\mu\left(S \setminus \bigcup_{B \in \mathcal{B}'} B\right) = 0.$$

Proof. First note that it suffices to prove the result for S bounded, say S is contained in some ball $\tilde{B}$. We may also suppose that each $B \in \mathcal{B}$ is a subset of $2\tilde{B}$.

Let $\mathcal{B}'$ be a disjoint sub-collection of $\mathcal{B}$ obtained from Proposition 3.7. Note that $\mathcal{B}'$ is countable. Indeed, for each $m \in \mathbb{N}$, at most $m\mu(2\tilde{B})$ balls $B \in \mathcal{B}'$ can satisfy $\mu(B) > 1/m$.

Enumerate $\mathcal{B}' = \{B_1, B_2, \ldots\}$. Since the B_i are disjoint subsets of $2\tilde{B}$,

$$\sum_{i>n} \mu(B_i) \to 0.$$

By the conclusion of Proposition 3.7,

$$S \setminus \bigcup_{i=1}^{n} B_i \subset \bigcup_{i>n} 5B_i$$

for each $n \in \mathbb{N}$. Since μ is doubling, $\mu(5B_i) \leq C_\mu^3 \mu(B_i)$ for each $i \in \mathbb{N}$ and so

$$\mu\left(S \setminus \bigcup_{i=1}^{n} B_i\right) \leq C \sum_{i>n} \mu(B_i) \to 0,$$

as required. □

Corollary 3.10. *Let μ be a doubling measure on X and $S \subset X$. Then for μ-a.e. $x \in S$,*

$$\frac{\mu(S \cap B(x,r))}{\mu(B(x,r))} \to 1 \quad \textit{as } r \to 0.$$

Such an x is called a density point *of x.*

See Exercise 3.8 for the proof.

Next we apply the covering theorems to Hausdorff measure.

Lemma 3.11. *Suppose that $s > 0$ and $A \subset X$ with $\mathcal{H}^s(A) < \infty$. Then*

$$2^{-s} \leq \Theta^{*,s}(A, x) \leq 1$$

for $\mathcal{H}^n$-a.e. $x \in A$.

Proof. The set of $x \in A$ with $\Theta^{*,s}(A,x) < 2^{-s}$ is a countable union countable of the sets

$$S_\delta := \{x \in A : \mathcal{H}^s(A \cap B(x,r)) < (1-\delta)r^s \ \forall 0 < r < \delta\}.$$

Thus, for the first inequality, it suffices to show that $\mathcal{H}^s(S_\delta) = 0$ for all $\delta > 0$.

Fix $\delta, \epsilon > 0$. We may cover S_δ by sets $E_1, E_2, \ldots$ such that, for each $i \in \mathbb{N}$, $\operatorname{diam} E_i < \epsilon$, $S_\delta \cap E_i \neq \emptyset$ and

$$\sum_{i \in \mathbb{N}} \operatorname{diam} E_i^s \leq \mathcal{H}^s(S_\delta) + \epsilon.$$

For each $i \in \mathbb{N}$ let $x_i \in S_\delta \cap E_i$ and set $r_i = \operatorname{diam} E_i$. Then

$$\begin{aligned}\mathcal{H}^s(S_\delta) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(S_\delta \cap E_i) &\leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(A \cap B(x_i, r_i)) \\ &\leq (1-\delta) \sum_{i \in \mathbb{N}} \operatorname{diam} E_i^s \leq (1-\delta)(\mathcal{H}^s(S_\delta) + \epsilon).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary and $\delta > 0$, this implies $\mathcal{H}^s(S_\delta) = 0$, as required.

For the second inequality, since $\mathcal{H}^s$ is Borel regular (see Exercise 3.1), it suffices to assume that A is Borel. As before, given $\delta > 0$, it suffices to prove that

$$S := \{x \in A : \Theta^{*,s}(A,x) > 1 + \delta\}$$

satisfies $\mathcal{H}^s(S) = 0$. Fix $\epsilon > 0$ and let $U \supset S$ be open with

$$\mathcal{H}^s(A \cap U) \leq \mathcal{H}^s(S) + \epsilon$$

(which exists by the outer regularity of the measure $\mathcal{H}^s|_A$). Let $\mathcal{B}_\epsilon$ be the collection of balls B centred at a point of S with $\operatorname{rad} B < \epsilon$ such that $B \subset U$ and

$$\mathcal{H}^s(A \cap B) > (1+\delta)(2 \operatorname{rad} B)^s. \tag{3.4}$$

This is a Vitali cover of S. Let $\mathcal{B}'_\epsilon$ be obtained from Proposition 3.7.

Since $\mathcal{H}^s(S) < \infty$, S is separable (see Exercise 3.6) and so $\mathcal{B}'_\epsilon = \{B_1, B_2, \ldots\}$ is countable and the conclusion of Proposition 3.7 states that

$$S \setminus \bigcup_{i \in \mathbb{N}} B_i \subset \bigcup_{i > n} 5B_i$$

for each $n \in \mathbb{N}$. Since $\operatorname{diam} B_i < \epsilon$ for each $i \in \mathbb{N}$, the B_i and $5B_i$ may be used to estimate $\mathcal{H}^s_{10\epsilon}(S)$. For each $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\mathcal{H}^s_{10\epsilon}(S) &\leq \sum_{i \in \mathbb{N}} (2 \operatorname{rad} B_i)^s + \sum_{i > n} (10 \operatorname{rad} B_i)^s \\ &\leq \sum_{i \in \mathbb{N}} \frac{\mathcal{H}^s(A \cap B_i)}{1+\delta} + 5^s \sum_{i > n} \frac{\mathcal{H}^s(A \cap B_i)}{1+\delta}\end{aligned}$$

where the second inequality follows by (3.4). Since the B_i are disjoint and $\mathcal{H}^s(A) < \infty$, the second term converges to 0 as $n \to \infty$. Since the B_i are subsets of U we obtain

$$\mathcal{H}^s_{10\epsilon}(S) \leq \frac{\mathcal{H}^s(A \cap U)}{1+\delta} \leq \frac{\mathcal{H}^s(S) + \epsilon}{1+\delta}.$$

Since $\epsilon > 0$ is arbitrary, this implies $\mathcal{H}^s(S) \leq \mathcal{H}^s(S)/(1+\delta)$ and hence $\mathcal{H}^s(S) = 0$, as required. □

Lemma 3.12. *Let X be a metric space, $s \geq 0$ and let $A \subset X$ be $\mathcal{H}^s$-measurable with $\mathcal{H}^s(A) < \infty$. Then*

$$\Theta^{*,s}(A,x) = 0$$

for $\mathcal{H}^s$-a.e. $x \notin A$.

Proof. It suffices to show that, for $t > 0$, the set

$$S = \{x \in X \setminus A : \Theta^{*,n}(A, x) > t\}$$

satisfies $\mathcal{H}^s(S) = 0$. Fix $\epsilon > 0$. Since A is $\mathcal{H}^s$-measurable, $\mathcal{H}^s|_A$ is Borel regular. Therefore, since $\mathcal{H}^s|_A(S) = 0$, there exists an open $U \supset S$ with

$$\mathcal{H}^s(A \cap U) = \mathcal{H}^s|_A(U) < \epsilon.$$

For each $x \in S$ and $\delta > 0$ there exists a ball B centred on x with $\operatorname{rad} B < \delta$ such that

$$\frac{\mathcal{H}^s(A \cap B)}{(2 \operatorname{rad} B)^s} > t.$$

By Lemma 3.5 there exists a disjoint collection $\mathcal{B}$ of such balls such that

$$S \subset \bigcup_{B \in \mathcal{B}} 5B.$$

Since $\mathcal{H}^s(A) < \infty$, A is separable and each of these balls contains a point of A, $\mathcal{B}$ is countable. Therefore

$$t\mathcal{H}^s_{5\delta}(S) \leq t \sum_{B \in \mathcal{B}} (2 \operatorname{rad} 5B)^s < 5^s \sum_{B \in \mathcal{B}} \mathcal{H}^s(A \cap B) \leq 5^s \mathcal{H}^s(A \cap U) < 5^s \epsilon.$$

Since $\delta, \epsilon > 0$ are arbitrary, this completes the proof. □

We also require a version of the coarea formula.

Lemma 3.13. *Let K be a compact metric space and $g\colon K \to Y$ Lipschitz. For any $s > 0$,*

$$\int_Y \operatorname{card}(g^{-1}(y)) \, \mathrm{d}\mathcal{H}^s(y) \leq \mathcal{H}^s(K).$$

Proof. Since K is compact,

$$f(y) = \operatorname{card}(g^{-1}(y))$$

is a Borel function. Indeed, for $\delta > 0$ define

$$f_\delta(y) = \max\{n \in \mathbb{N} : \exists x_1, \ldots, x_n \in g^{-1}(y) \text{ with } \|x_i - x_j\| \geq \delta \; \forall 1 \leq i \neq j \leq n\}.$$

Then f_δ monotonically increases to f as $\delta \to 0$. Since K is compact, the f_δ are lower semi-continuous and hence f is Borel. Moreover, by the monotone convergence theorem, it suffices to bound the integral of each f_δ.

Fix $\delta > 0$ and decompose K into disjoint sets $E_1, E_2, \ldots$ with $\operatorname{diam} E_i < \delta$. Then

$$f_\delta(y) \leq \operatorname{card}(\{i : E_i \cap g^{-1}(y) \neq \emptyset\}).$$

Therefore

$$\begin{aligned}
\int_Y f_\delta \, \mathrm{d}\mathcal{H}^s &\leq \int_Y \sum_{i \in \mathbb{N}} \chi_{\{(i,y) : E_i \cap g^{-1}(y) \neq \emptyset\}} \, \mathrm{d}\mathcal{H}^s \\
&= \sum_{i \in \mathbb{N}} \int_Y \chi_{\{(i,y) : E_i \cap g^{-1}(y) \neq \emptyset\}} \, \mathrm{d}\mathcal{H}^s \\
&\leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(g(E_i^s)) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(E_i^s) = \mathcal{H}^s(K)
\end{aligned}$$

as required. □

The final result we require in order to prove Theorem 3.2 is the folowing special case of the Lusin-Novikov theorem from descriptive set theory, see [15, Exercise 18.14].

Lemma 3.14. *Let K and Y be compact metric spaces and $f\colon K \to Y$ Borel. Suppose that* card $f^{-1}(y) < \infty$ *for every $y \in Y$. Then there exists a Borel function $g\colon f(K) \to K$ such that $K' := g(f(K))$ is Borel and $f(g(y)) = y$ for all $y \in f(K)$.*

Proof. First note that, for each $n \in \mathbb{N}$,

$$Y'_n := \{y \in Y : \operatorname{card} f^{-1}(y) \le n\}$$

is Borel. Indeed, for $n = 0$ this is immediate and for $n \ge 1$ we have

$$Y'_n = \bigcup_{k \in \mathbb{N}} \{y \in Y : \exists x_1, \ldots, x_n \in f^{-1}(y),\ d(x_i, x_j) \ge \frac{1}{k}\ 1 \le i \ne j \le n\}.$$

Since f is continuous and K is compact, each term in this union is a closed subset of $f(K)$ and hence Y'_n is Borel. Consequently,

$$Y_n := \{y \in Y : \operatorname{card} f^{-1}(y) = n\}$$

is also Borel, as are the closed subsets

$$Y_n^k := \{y \in Y : \operatorname{card} f^{-1}(y) = n,\ d(x, x') \ge \frac{1}{k}\ \forall x \ne x' \in f^{-1}(y)\}.$$

Moreover, $f(K)$ is covered by the countable union of the Y_n^k, $k, n \in \mathbb{N}$.

For fixed $k, n \in \mathbb{N}$ let $B_1, \ldots, B_N$ be balls of radius $1/k$ that cover K. Then the sets

$$Y_{n,i}^k := \{y \in Y_n^k : \operatorname{card} f^{-1}(y) \cap B_i = 1\}$$

cover Y_n^k and are closed. Define $g\colon f(K) \to K$ on each $Y_{n,i}^k$ by mapping y to the unique element of $f^{-1}(y) \cap B_i$. Then g is continuous on each $Y_{n,i}^k$ and hence Borel.

Finally, $g(f(K))$ is a countable union of sets of the form $f^{-1}(Y_{n,i}^k) \cap B_i$, each of which are Borel. □

Proof of Theorem 3.2. First let $V \subset X$ be Borel and note that (3.1) holds for the function $f|_V$ and for $\mathcal{H}^s$-a.e. $x \in V$. Indeed, for $\mathcal{H}^s$-a.e. $x \in V$, Lemma 3.12, $\Theta^{*,s}(X \setminus V, x) = 0$ and for such an x,

$$\begin{aligned}\limsup_{r \to 0} \frac{\mathcal{H}^s(B(f(x), \lambda_x r) \setminus f(V \cap B(x,r)))}{(2\lambda_x r)^s} &\le \limsup_{r \to 0} \frac{\mathcal{H}^s(B(f(x), \lambda_x r) \setminus f(B(x,r)))}{(2\lambda_x r)^s} \\ &\quad + \limsup_{r \to 0} \frac{L^s \mathcal{H}^s(B(x,r) \setminus V)}{(2\lambda_x r)^s} \\ &< \frac{1}{2}\Theta_*^s(Y, f(x)) + \frac{L^s}{\lambda_x^s}\Theta^{*,s}(X \setminus V, x). \qquad (3.5)\end{aligned}$$

A consequence of (3.5) is

$$\mathcal{H}^s(V) > 0 \Rightarrow \mathcal{H}^s(f(V)) > 0. \qquad (3.6)$$

Now suppose $\mathcal{H}^s(V) > 0$. By Lemma 3.13, for $\mathcal{H}^s$-a.e. $y \in Y$, $\operatorname{card} f^{-1}(y) < \infty$. Hence, by (3.6), for $\mathcal{H}^s$-a.e. $x \in V$,

$$\operatorname{card}\{x' \in V : f(x') = f(x)\} < \infty. \qquad (3.7)$$

Let $K \subset V$ be a positive measure Borel set for which (3.7) holds for all $x \in K$. By Lemma 3.14, there exists a Borel function $g\colon f(K) \to K$ such that $V' := g(f(K))$ is Borel and $f(g(y)) = y$ for all $y \in f(K)$. By (3.6), $\mathcal{H}^s(f(V')) = \mathcal{H}^s(f(K)) > 0$ and hence, since f is Lipschitz, $\mathcal{H}^s(V') > 0$.

Note that if $x \in X$ satisfies (3.1), then it also satisfies (3.1) for all $0 < \lambda \le \lambda_x$. For $i \in \mathbb{N}$ let $1 \ge \lambda_i \searrow 0$ and define S_i to be the set of $x \in V'$ for which

$$\sup_{0 < r < \lambda_i} \frac{\mathcal{H}^s(B(f(x), \lambda_i r) \setminus f(V' \cap B(x,r)))}{(2\lambda_i r)^s} < \inf_{0 < r' < \lambda_i} \frac{1}{2}(1 - \lambda_i)^s \frac{\mathcal{H}^s(B(f(x), r'))}{(2r')^s}.$$

Then, by (3.5), the S_i monotonically increase to a full measure subset of V'. Therefore, there exist $i \in \mathbb{N}$ and $S' \subset S_i$ with $\mathcal{H}^s(S') > 0$ and $\operatorname{diam} S' \le \lambda_i$. For any $0 < r < \lambda_i$, setting $r' = (1 - \lambda_i^2)\lambda_i r$ shows that $f|_{V'}$ satisfies $D(\lambda_i, (1 - \lambda_i^2)\lambda_i, (1 + \lambda_i)^{-s})$

on S'. Since $f|_{V'}$ is injective, (3.2) holds for all $x, y \in S'$ and hence the Lemma implies that $f|_{S'}$ is bi-Lipschitz.

The bi-Lipschitz condition extends to the closure of S'. Hence $\overline{S'} \cap V$ is a Borel subset of V of positive measure on which f is bi-Lipschitz. Since V is an arbitrary Borel subset of positive measure, the conclusion follows by Exercise 3.5. □

Remark 3.15. That the converse to Theorem 3.2 is true. Namely, if X is n-rectifiable then there exist *countably many* Lipschitz functions f and $R > 0$ for which (3.1) holds with $\lambda = 1$ (with respect to some norm). Indeed, the γ_i from Corollary 1.13 are such functions. If the rectifiable set is a subset of Euclidean space, finitely many Lipschitz functions suffice (and may be chosen to be the orthogonal projection onto the span of n coordinate axes).

Theorem 3.2 is the starting point of the theory of rectifiability in metric spaces, leading to characterisations given in [5, 6, 7]. Recall that a subset of a complete metric space is *residual* if it contains a countable intersection of open dense sets. A $\mathcal{H}^n$-measurable subset S of a metric space X is *purely n-unrectifiable* if $\mathcal{H}^n(S \cap E) = 0$ for all n-rectifiable $E \subset X$.

Theorem 3.16 (B. [6], B, Orponen, Weigt [4])**.** *Let X be a complete metric space and $S \subset X$ purely n-unrectifiable with $\mathcal{H}^n(S) < \infty$. The set of all 1-Lipschitz $f \colon X \to \mathbb{R}^n$ with $\mathcal{H}^n(f(S)) = 0$ is residual. Conversely, if $E \subset X$ is n-rectifiable with $\mathcal{H}^n(E) > 0$, the set of 1-Lipschitz $f \colon X \to \mathbb{R}^n$ with $\mathcal{H}^n(f(E)) > 0$ is residual.*

This may be seen as a counterpart to the Besicovitch–Federer projection theorem [18, Theorem 18.1]. Recent work with Takáč [8] shows that the converse statement may be improved under hypotheses on X; on the other hand, in some circumstances it is impossible to improve the converse statement.

3.1. **Exercises.**

Exercise 3.1. *Let X be a metric space and $s \geq 0$. A measure μ on X is* Borel regular *if for every $S \subset X$ there exists a Borel $B \supset S$ with $\mu(B) = \mu(S)$.*

(1) *Show that $\mathcal{H}^s$ is Borel regular. Hint: first show that in the definition of $\mathcal{H}^s$, we may take F to be the collection of* closed *sets.*
(2) *We are usually interested in $\mathcal{H}^s|_A$ for some $A \subset X$. Show that for* any *$A \subset X$, $\mathcal{H}^s|_A$ is a Borel measure.*
(3) *Now assume that $A \subset X$ is $\mathcal{H}^s$-measurable with $\mathcal{H}^s(A) < \infty$. Show that $\mathcal{H}^s|_A$ is Borel regular. Hint: show that there exist Borel sets $B \supset A \supset B'$ with $\mathcal{H}^s(B \setminus B') = 0$.*

Exercise 3.2. *Let X be a separable metric space. Show that for any collection of balls, there exists a maximal disjoint sub-collection.*

Exercise 3.3. *Show that the $5r$ covering Lemma may not be true if the radii are not uniformly bounded.*

Exercise 3.4. *Let μ be a finite Borel measure on a metric space X. Prove that for every Borel $B \subset X$,*

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ closed}\} \tag{3.8}$$

and

$$\mu(B) = \inf\{\mu(U) : U \supset B \text{ open}\}. \tag{3.9}$$

Property (3.8) *is called* inner regularity by closed sets *and* (3.9) *is called* outer regularity by open sets.

Hint: observe that it suffices to show that all Borel sets satisfy (3.8)*. Show that the set*

$$\{B \subset X : B \text{ and } X \setminus B \text{ satisfy } (3.8)\}$$

is a σ-algebra that contains all closed subsets of X.

Show that a σ-finite measure μ is inner regular by closed sets. Show that a σ-finite measure μ is outer regular by open sets if there exist open sets $U_i \subset X$ with $\mu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $X = \bigcup_{i\in\mathbb{N}} U_i$. Give an example of a σ-finite μ that is not *outer regular by open sets.*

Prove that a σ-finite Borel regular measure on a complete and separable metric space is inner regular by compact *sets.*

Exercise 3.5. *Let μ be a finite measure on a set X. Let $\mathcal{S}$ be a collection of μ-measurable subsets of X such that, for each μ-measurable $S \subset X$ of positive measure, there exists $S' \in \mathcal{S}$ with $\mu(S') > 0$ and $S' \subset S$. Show that there exists $S_i \in \mathcal{S}$ with*

$$\mu\left(X \setminus \bigcup_{i\in\mathbb{N}} S_i\right) = 0.$$

Exercise 3.6. *For $s \geq 0$ let X be a metric space with $\mathcal{H}^s(X) < \infty$. Show that X is separable.*

Exercise 3.7. *Extend Lemma 3.13 to the case that $g\colon X \to Y$ with X complete and separable.*

Exercise 3.8.
(1) *Suppose that μ is a doubling measure on X and $B \subset X$ is Borel. Use Theorem 3.9 to show that*

$$\lim_{r\to 0} \frac{\mu(B \cap B(x,r))}{\mu(B(x,r))} = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

for μ-a.e. $x \in X$.

(2) *Suppose that $S \subset X$ (not necessarily μ-measurable) and that $B \supset S$ is μ-measurable with $\mu(S) = \mu(B)$. Show that $\mu(S \cap M) = \mu(B \cap M)$ for all μ-measurable $M \subset X$.*

(3) *Suppose that μ is an doubling Borel regular measure and that $S \subset X$. Show, even if S is not μ-measurable, that*

$$\frac{\mu(S \cap B(x,r))}{\mu(B(x,r))} \to 1 \quad \text{as } r \to 0$$

for μ-a.e. $x \in S$.

4. Convergence of metric measure spaces

Classically, rectifiable subsets of Euclidean space are characterised by the existence of an *approximate tangent plane* at almost every point. More precisely, let $E \subset \mathbb{R}^m$ satisfy $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable if and only if, for $\mathcal{H}^n$-a.e. $x \in E$, there exists an affine subspace $V_x \subset \mathbb{R}^m$ such that

$$\frac{\mathcal{H}^n(E \cap B(x,r) \setminus V_x)}{r^n} \to 0 \tag{4.1}$$

as $r \to 0$. See [18, Theorem 15.19]. Stronger statements implying rectifiability are given by Besicovitch [9], Marstrand [17] and Mattila [19] by allowing the approximating tangent plane to depend on r; the tangent may rotate as one zooms into E. See [18, Theorem 16.2].

Analogous statements hold for rectifiable subsets of a metric space. However, in order to state the results, we must understand what we mean by a tangent in this setting. Fundamentally, this requires us to define a notion of convergence of metric measure spaces.

4.1. **Convergence of metric measure spaces.**

Definition 4.1. For a complete metric space Z define $C_{\mathrm{bs}}(Z)$ to be the set of bounded and continuous $g\colon Z \to \mathbb{R}$ with bounded support. Also let $\mathcal{M}_{\mathrm{loc}}(Z)$ denote the set of Borel regular measures on Z that are finite on bounded sets.

We say that $\mu_i \in \mathcal{M}_{\mathrm{loc}}(Z)$ converges to $\mu \in \mathcal{M}_{\mathrm{loc}}(Z)$, written $\mu_i \to \mu$, if

$$\int_Z g \, \mathrm{d}\mu_i \to \int_Z g \, \mathrm{d}\mu$$

for all $g \in C_{\mathrm{bs}}(Z)$.

Definition 4.2. A *pointed metric measure space* (X, d, μ, x) consists of a complete and separable metric space (X, d), a Borel regular measure $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ and a distinguished point $x \in \operatorname{spt} \mu$.

Inspired by the notion of Gromov–Hausdorff convergence (defined below), we can consider convergence of pointed metric measure spaces.

Definition 4.3. A sequence (X_i, d_i, μ_i, x_i) of pointed metric measure spaces converges to a pointed metric measure space (X, d, μ, x) if there exists a complete metric space Z and isometric embeddings $X_i \hookrightarrow Z$, $X \hookrightarrow Z$ such that $x_i \to x$ and $\mu_i \to \mu$ in Z.

An point that will become very useful later is that this convergence can be metrised.

Fact. *There exists a separable metric d_* on the set of isometry classes of all pointed metric measure spaces that metrises the convergence in Definition 4.3.*

In this section we will prove the following theorem.

Theorem 4.4. *Let (X_i, d_i, μ_i, x_i) be a sequence of uniformly doubling metric measure spaces such that*

$$\sup_i \mu_i(B(x_i, 1)) < \infty. \tag{4.2}$$

There exists a pointed metric measure space (X, d, μ, x) such that (after possibly taking a subsequence),

$$(X_i, d_i, \mu_i, x_i) \to (X, d, \mu, x).$$

4.2. **Hausdorff distance.** In order to prove Theorem 4.4, we must first discuss the convergence of metric spaces. To do this, we first consider the convergence of subsets of a fixed metric space.

Definition 4.5. Let X be a metric space. For $C, D \subset X$, define the *Hausdorff distance* between C and D as

$$\mathrm{d_H}(C, D) = \inf\{r > 0 : B(C, r) \supset D,\ B(D, r) \supset C\}.$$

Lemma 4.6. *For any metric space X, $\mathrm{d_H}$ satisfies the triangle inequality on the power set of X. Moreover, $\mathrm{d_H}$ is a metric on $\mathcal{C}_{\mathrm{b}}(X)$, the set of non-empty closed and bounded subsets of X.*

Proof. Let $C, D, E \subset X$ and, for $r, r' > 0$, suppose

$$B(C, r) \supset D,\ B(D, r) \supset C,\ B(D, r') \supset E,\ B(E, r') \supset D. \tag{4.3}$$

Then, by the triangle inequality on X,

$$B(C, r + r') \supset E \text{ and } B(E, r + r') \supset C.$$

Indeed, if $e \in E$ then there exist $d \in D$ with $d(d, e) \leq r'$ and $c \in C$ with $d(c, d) \leq r$. Thus $d(c, e) \leq r + r'$. Taking the infimum over all $r, r' > 0$ satisfying (4.3) gives the triangle inequality for $\mathrm{d_H}$.

To see that $\mathrm{d_H}$ is a metric on $\mathcal{C}_\mathrm{b}(X)$, suppose that $C, D \in \mathcal{C}_\mathrm{b}(X)$, $\mathrm{d_H}(C, D) = 0$ and $x \in D$. We need to show $x \in C$. Since $D \subset B(C, r)$ for every $r > 0$, there exist $x_n \in C$ with $x_n \to x$. Since C is closed, $x \in C$, as required. Finally, since the elements of $\mathcal{C}_\mathrm{b}(X)$ are non-empty and bounded, $\mathrm{d_H}$ is finite on $\mathcal{C}_\mathrm{b}(X)$. □

Proposition 4.7. *If X is a complete metric space, then $(\mathcal{C}_\mathrm{b}(X), \mathrm{d_H})$ is also complete.*

Proof. Let C_n be a Cauchy sequence in $\mathcal{C}_\mathrm{b}(X)$. As usual, by taking a subsequence if necessary, we may suppose that $\mathrm{d_H}(C_n, C_m) < 2^{-n}$ for each $m \geq n \in \mathbb{N}$. First fix $x_1 \in C_1$ and let $x_2 \in C_2$ be such that $d(x_1, x_2) < 2^{-1}$. Proceeding inductively, we select $x_n \in C_n$ with $d(x_{n-1}, x_n) < 2^{-n}$ for each $n \in \mathbb{N}$. Then x_n is a Cauchy sequence in X and hence converges to some $x \in X$. Now define

$$D_n = \overline{\bigcup_{k \geq n} C_k},$$

a closed subset of X. Since $x_k \in D_n$ for all $k \geq n$, $x \in D_n$ for all $n \in \mathbb{N}$. Therefore, $C := \cap_n D_n$ is closed and non-empty.

Since $\mathrm{d_H}(C_n, C_m) < 2^{-n}$ for each $m \geq n \in \mathbb{N}$,

$$C \subset D_n \subset B(C_n, 2^{-n})$$

for each $n \in \mathbb{N}$. On the other hand, for $m \geq n$,

$$C_n \subset B(C_m, 2^{-n}) \subset B(D_m, 2^{-n}) \subset B(C, 2^{1-n}).$$

Thus $\mathrm{d_H}(C, C_n) \to 0$, as required. □

Theorem 4.8 (Blaschke)**.** *If X is totally bounded then $\mathcal{C}_\mathrm{b}(X)$ is too. In particular, if X is compact then $\mathcal{C}_\mathrm{b}(X)$ is.*

Proof. For $\epsilon > 0$ let $F \subset X$ be finite such that $B(F, \epsilon) \supset X$ and let $\mathcal{F}$ be the set of non-empty subsets of F. Then $B(\mathcal{F}, \epsilon) = \mathcal{C}_\mathrm{b}(X)$ (see Exercise 4.3) and hence $\mathcal{C}_\mathrm{b}(X)$ is totally bounded. □

4.3. **Gromov–Hausdorff convergence.** Gromov [12, 13] introduced the idea of comparing different metric spaces by first embedding them into a common metric space and taking the Hausdorff distance between their images. This induced a metric, the *Gromov–Hausdorff distance* between two bounded metric spaces. For our discussion, we only require the notion of convergence defined by this metric, which we now state.

Definition 4.9. A sequence X_i of complete and bounded metric spaces *Gromov–Hausdorff converges* to a metric space X if there exists a metric space Z and isometric embeddings $X_i \hookrightarrow Z$, $X \hookrightarrow Z$ such that $X_i \to X$ with respect to the Hausdorff distance in X.

A key result of Gromov is the following compactness theorem [12].

Definition 4.10. A collection $\mathcal{M}$ of metric spaces is *uniformly totally bounded* if

$$\sup\{\operatorname{diam} X : X \in \mathcal{M}\} < \infty$$

and, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that each $X \in \mathcal{M}$ is covered by N balls of radius ϵ.

Compare the proof of the following proposition to the Kuratowski embedding given in Lemma 1.8.

Proposition 4.11. *Let N_i be a sequence of natural numbers and let $\mathcal{S}$ be the set of all sequences $n \in \mathbb{N}^{\mathbb{N}}$ such that $n_i \leq N_i$ for each $i \in \mathbb{N}$. Let $D > 0$, $\epsilon_i = 2^{-i}$ and let $\mathcal{S}'$ be the set of all $f\colon \mathcal{S} \to \mathbb{R}$ with $\|f\|_\infty \leq D$ and the following property: If $n, n' \in \mathcal{S}$ are such that $n_k = n'_k$ for all $1 \leq k \leq i$, then $|f(n) - f(n')| \leq 2\epsilon_i$. Then $\mathcal{S}'$ is compact.*

Moreover, suppose that X is a metric space with diam $X \leq D$ *such that, for each $i \in \mathbb{N}$, there exist $x_1, \ldots, x_{N_i} \in X$ with*

$$X = \bigcup_{k=1}^{N_i} B(x_k, \epsilon_i).$$

Then there exists an isometric embedding $X \hookrightarrow \mathcal{S}'$.

Proof. Certainly $\mathcal{S}'$ is a closed subset of all bounded functions on $\mathcal{S}$ and hence is complete. It is also totally bounded: for any $i \in \mathbb{N}$, the set of functions mapping each n_k with $1 \leq k \leq i$ to an element of $\epsilon_i\{0, \ldots, \lceil D/\epsilon_i \rceil\}$ (and all other n_k to 0) is a (finite) $2\epsilon_i$-net in $\mathcal{S}'$.

We encode any $n \in \mathcal{S}$ as a point in X as follows. First set $y_1 = x_{n_1}$. If $d(x_{n_2}, y_1) \leq \epsilon_1$ then set $y_2 = x_{n_2}$. Otherwise set $y_2 = y_1$. We repeat this process iteratively: for each $i \in \mathbb{N}$, if $d(x_{n_i}, y_{i-1}) \leq \epsilon_{i-1}$ then set $y_i = x_{n_i}$, otherwise set $y_i = y_{i-1}$. Then y_i forms a Cauchy sequence in X, which converges to some y. We set $e(n) = y$. By hypothesis, e is a surjection.

For $x \in X$ now define $\iota(x)\colon \mathcal{S} \to \mathbb{R}$ by

$$\iota(x)(n) = d(x, e(n)),$$

so that $\|\iota(x)\| \leq$ diam $X \leq D$. Moreover, by construction, if $n, n' \in \mathcal{S}$ are such that $n_i = n'_i$ for all $1 \leq i \leq k$ then

$$d(e(n), e(n')) \leq 2\epsilon_i.$$

Consequently, by the triangle inequality,

$$|\iota(x)(n) - \iota(x)(n')| \leq 2\epsilon_i$$

and so $\iota(x) \in \mathcal{S}'$. Also by the triangle inequality, ι is 1-Lipschitz. Finally, if $x, y \in X$ and $n \in \mathcal{S}$ is such that $e(n) = x$, then

$$|\iota(x)(n) - \iota(y)(n)| = |d(x, x) - d(y, x)|,$$

and so ι is an isometry. □

Theorem 4.12. *Let X_i be a uniformly totally bounded sequence of compact metric spaces. Then there exists a compact metric space X such that (after possibly passing to a subsequence) X_i Gromov–Hausdorff converges to X.*

Proof. Combine Proposition 4.11 and Theorem 4.8 □

Proof of Theorem 4.4. First fix $R \in \mathbb{N}$ and consider the sequence

$$(B(x_i, R), d_i, (\mu_i)|_{B(x_i,R)}, x_i).$$

By Exercise 4.5 the Y_i are uniformly totally bounded. By Theorem 4.12, after passing to a subsequence, there exists a compact metric space Z and isometric embeddings $Y_i \hookrightarrow Z$ such that $Y_i \to X$ in Z. By taking a further subsequence, we may suppose x_i converges to some $x \in X$.

By (4.2) and the uniform doubling property, $\mu_i(Y_i)$ is uniformly bounded. Therefore, by the Riesz representation theorem, we may pass to another subsequence and suppose that $(\mu_i)|_{B(x_i,R)} \to \mu$ in $\mathcal{M}_{\text{loc}}(Z)$.

The final step is to repeat the first part of the argument, for each $R \in \mathbb{N}$, on the spaces $B(x_i, R)$, and take a diagonal subsequence (this requires the fact that this convergence is metrisable). □

4.4. **Exercises.**

Exercise 4.1. *Show that the Hausdorff distance between any set and its closure is zero.*

Exercise 4.2. *Show that X isometrically embeds into $\mathcal{C}_{\mathrm{b}}(X)$ and that it's image is closed. Hence show that the converses to Proposition 4.7 and Theorem 4.8 are true.*

Exercise 4.3. *Complete the proof of Theorem 4.8.*

Exercise 4.4. *Show that $\mathcal{S} \subset (\mathcal{K}, \mathrm{d}_{\mathrm{GH}})$ is totally bounded if and only if it is uniformly totally bounded.*

Exercise 4.5. *Let (X, d, μ) be a C-doubling metric space. Show that there exists $N \in \mathbb{N}$, depending only upon C, such that each ball $B(x, r)$ in X may be covered by N balls of radius $r/2$. That is X is a* doubling *metric space.*

5. Tangent spaces of pointed metric measure spaces

Definition 5.1. For $(X, d, \mu, x) \in \mathcal{M}_{\mathrm{loc}}$ and $r > 0$ let

$$T_r(X, d, \mu, x) = \left(X, \frac{d}{r_i}, \frac{\mu}{\mu(B(x, r_i))}, x\right).$$

A *tangent space* of (X, d, μ, x) is any $(Y, \rho, \nu, y) \in \mathcal{M}_{\mathrm{loc}}$ for which there exist $r_i \to 0$ such that

$$T_r(X, d, \mu, x) \to (Y, \rho, \nu, y).$$

We write $\mathrm{Tan}(X, \mu, x)$ for the set of all tangent spaces to (X, μ, x).

Proposition 5.2. *Let X be a metric space and let $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ be doubling. For μ-a.e. $x \in X$ and $R > 0$,*

$$\{T_r(\mu, x) : 0 < r < R\} \text{ is pre-compact.} \tag{5.1}$$

In particular, $\mathrm{Tan}(\mu, x)$ *is a non-empty compact metric space when equipped with d_* and*

$$\forall \delta > 0,\ \exists r_x > 0 \text{ s.t. } d_*(T_r(\mu, x), \mathrm{Tan}(\mu, x)) \leq \delta\ \forall 0 < r < r_x. \tag{5.2}$$

Proof. First note that, for any $r > 0$,

$$\frac{\mu(B_{d/r}(x, R))}{\mu(B_d(x, r))} = \frac{\mu(B_d(x, Rr))}{\mu(B_d(x, r))} \leq C_\mu^{4 \log_2 R}.$$

Here the subscripts on the balls indicate the metrics used to define the balls. Thus

$$\{\nu(B(x, R)) : (\nu, x) \in T_r(\mu, x),\ 0 < r < R\}$$

is bounded and the measures in $T_r(\mu, x)$ are uniformly doubling. Thus Theorem 4.4 implies (5.1).

By applying (5.1) to an arbitrary sequence $r_i \to 0$ we see that $\mathrm{Tan}(\mu, x)$ is non-empty. To see that it is compact, for each $j \in \mathbb{N}$ let $(\nu_j, y_j) \in \mathrm{Tan}(\mu, x)$ and let $0 < r_j < 1/j$ be such that

$$d_*(T_{r_j}(\mu, x), (\nu_j, y_j)) < 1/j.$$

By (5.1) there exists a subsequence $r_{j_k} \to 0$ and a $(\nu, y) \in \mathcal{M}$ such that

$$d_*(T_{r_{j_k}}(\mu, x), (\nu, y)) \to 0.$$

In particular, $(\nu, y) \in \mathrm{Tan}(\mu, x)$ and, by the triangle inequality, $(\nu_{j_k}, y_{j_k}) \to (\nu, y)$, as required.

Finally, given $\delta > 0$, the existence of such an r_x is given by the contrapositive to (5.1). □

Tangent measures were first defined, for measures on a fixed Euclidean space, by Preiss [20]. Those tangent measures satisfy many important properties that we now show also hold in our more general setting.

Lemma 5.3. *Let X be a metric space, $\mu \in \mathcal{M}_{\text{loc}}(X)$ and $x \in \operatorname{spt}\mu$. Suppose that (Y, ν, y) is a pointed metric measure space and $r_k \to 0$ is such that*

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, x\right), (\nu, y)\right) \to 0.$$

Suppose that x is a density point of $E \subset X$. Then for any $b \in \operatorname{spt}\nu$, there exists $b_k \in E$ such that

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, b_k\right), (\nu, b)\right) \to 0.$$

Proof. Let (Z, ζ) be a complete and separable metric space for which there exist isometric embeddings

$$(\operatorname{spt}\mu, \frac{d}{r_k}, x), (\operatorname{spt}\nu, y) \hookrightarrow (Z, \zeta)$$

such that (writing μ_k and x_k for the isometric copy of $\mu \in \mathcal{M}_{\text{loc}}(X, \frac{d}{r_k})$ and x respectively) $x_k \to y$ and

$$\frac{\mu_k}{\mu(B(x,r_k))} \to \nu_x.$$

Also let E_k be the isometric copy of $(E, \frac{d}{r_k})$.

Now let $b \in \operatorname{spt}\nu$ and set $R = 2\zeta(y, b)$. Since x is a density point of E,

$$\lim_{k\to\infty} \frac{\mu_k(E_k \cap B(x_k, R))}{\mu_k(B(x_k, R))} = \lim_{k\to\infty} \frac{\mu(E \cap B(x, Rr_k))}{\mu(B(x, Rr_k))} = 1. \tag{5.3}$$

We claim that, for any $b \in \operatorname{spt}\nu$, there exists $b_k \in E_k$ with $\zeta(b, b_k) \to 0$. Indeed, if not, there exists $\delta > 0$ such that $B(b, \delta) \cap E_k = \emptyset$ for infinitely many k. Exercise 5.1 then implies

$$\begin{aligned}\liminf_{k\to\infty} \frac{\mu_k(E_k \cap B(x_k, R))}{\mu_k(B(x_k, R))} &\leq 1 - \limsup_{k\to\infty} \frac{\mu_k(B(b,\delta))}{\mu_k(B(x_k, R))} \\ &\leq 1 - \frac{\nu(B(b, \frac{\delta}{2}))}{\nu(B(y, 2R))} < 1,\end{aligned}$$

since $b \in \operatorname{spt}\nu$. This contradicts (5.3) and so the claim holds. □

Proposition 5.4. *Let (X, d, μ) be doubling.*

(1) *For any $x \in X$ and $s > 0$,*

$$(Y, \rho, \nu, y) \in \operatorname{Tan}(X, d, \mu, x) \Rightarrow (Y, s\rho, \nu, y) \in \operatorname{Tan}(X, d, \mu, x).$$

(2) *For μ-a.e. $x \in X$,*

$$\begin{aligned}(Y, \rho, \nu, y) \in \operatorname{Tan}(X, d, \mu, x) \text{ and } (Z, \zeta, \lambda, z) \in \operatorname{Tan}(Y, \rho, \nu, y) \\ \Rightarrow (Z, \zeta, \lambda, z) \in \operatorname{Tan}(X, d, \mu, x).\end{aligned}$$

(3) *For μ-a.e. $x \in X$,*

$$(Y, \rho, \nu, y) \in \operatorname{Tan}(X, d, \mu, x) \text{ and } z \in \operatorname{spt}\nu \Rightarrow (Y, \rho, \nu, z) \in \operatorname{Tan}(X, d, \mu, x).$$

Proof. The first point simply follows from appropriately scaling the embeddings in the definition of $\operatorname{Tan}(X, d, \mu, x)$. The second point then follows from the first and Proposition 5.2. The proof of the third point depends on the existence of d_*.

For $\eta > 0$ let A_η be the set of $x \in \operatorname{spt}\mu$ for which there exist $(\nu_x, y_x) \in \operatorname{Tan}(\mu, x)$ and $b_x \in \operatorname{spt}\nu_x$ such that

$$d_*((\nu_x, b_x), T_r(\mu, x)) > \eta \quad \forall 0 < r < \eta.$$

Note that, if $x \notin \cup_{i\in\mathbb{N}} A_{1/i}$, then x satisfies the required conclusion. We suppose that $\mu(A_\eta) > 0$ for some $\eta > 0$ and aim for a contradiction. Since d_* is separable, there exists $E \subset A_\eta$ with $\mu(E) > 0$ such that

$$d_*((\nu_x, b_x), (\nu_{x'}, b_{x'})) < \frac{\eta}{2} \tag{5.4}$$

for every $x, x' \in E$.

By Corollary 3.10 there exists a density point x of E. By Lemma 5.3, there exist $b_k \in E$ be such that

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, b_k\right), (\nu_x, b_x)\right) \to 0.$$

Let $k \in \mathbb{N}$ satisfy $r_k < \eta$ and

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, b_k\right), (\nu_x, b_x)\right) < \frac{\eta}{2}.$$

However, since $b_k \in E$,

$$\begin{aligned}\eta &< d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, b_k\right), (\nu_{b_k}, b_{b_k})\right)\\ &\leq d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, b_k\right), (\nu_x, b_x)\right) + d_*((\nu_x, b_x), (\nu_{b_k}, b_{b_k}))\\ &\leq \frac{\eta}{2} + \frac{\eta}{2},\end{aligned}$$

using (5.4) for the final inequality. This contradiction implies that we must have $\mu(A_\eta) = 0$ for all $\eta > 0$. □

Remark 5.5. The above theory extends to tangents of *asymptotically* doubling metric spaces, which satisfy the weaker hypothesis

$$\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$$

for μ-a.e. $x \in X$. In order to establish this, minor technical modifications to the statement and proof of Theorem 4.4 are required. Once this is obtained, Propositions 5.2 and 5.4 follow as above. By Theorem 1.15, Hausdorff measure restricted to a rectifiable set is asymptotically doubling.

We now return to rectifiable sets. A consequence of Theorem 1.15 is the following.

Theorem 5.6. *Let (X, d) be a complete metric space, $n \in \mathbb{N}$ and $E \subset X$ an n-rectifiable set with $\mathcal{H}^n(E) < \infty$. For $\mathcal{H}^n$-a.e. $x \in E$,*

$$\operatorname{Tan}(X, d, \mathcal{H}^n|_E, x) = \{(\mathbb{R}^n, \|\cdot\|_x, \mathcal{H}^n/2^n, 0)\}$$

as $r \to 0$.

In fact, rectifiable metric spaces can be characterised via tangent spaces.

Theorem 5.7 ([5])**.** *Let (X, d) be a complete metric space, $n \in \mathbb{N}$ and $E \subset X$ a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(E) < \infty$. Suppose that, for $\mathcal{H}^n$-a.e. $x \in E$, $\Theta^n_*(E, x) > 0$ and there exists a $K_x \geq 1$ such that each element of* $\operatorname{Tan}(X, d, \mathcal{H}^n|_E, x)$ *is supported on a K_x-bi-Lipschitz image of $\mathbb{R}^n$. Then E is n-rectifiable.*

5.1. Exercises.

Exercise 5.1. *Suppose that $\mu_i \to \mu$ in $\mathcal{M}_{\mathrm{loc}}(Z)$ and $x_i \to x$. Show that*

$$\mu(U(x,r)) \leq \liminf_{i\to\infty} \mu_i(U(x_i,r)) \tag{5.5}$$

and

$$\mu(B(x,r)) \geq \limsup_{i\to\infty} \mu_i(B(x_i,r)). \tag{5.6}$$

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