# GRADUATE REAL ANALYSIS

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## Part 1. General measure theory

### 1. Measures

We wish to assign a value to the size of subsets of some given space, such as the length, area or volume of subsets of $\mathbb{R}^m$.

**Definition 1.1.** A *measure* $\mu$ on a set $X$ is a function

$$\mu\colon \{A : A \subset X\} \to [0, \infty]$$

such that

(1) $\mu(\emptyset) = 0$;
(2) $\mu(A) \subset \mu(B)$ whenever $A \subset B \subset X$;
(3) $\mu(\bigcup_{i\in\mathbb{N}} A_i) \le \sum_{i\in\mathbb{N}} \mu(A_i)$ whenever $A_1, A_2, \ldots \subset X$.

A function satisfying (2) is said to be *monotonic* and a function satisfying (3) is said to be *countably sub-additive*.

**Definition 1.2.** Let $\mu$ be a measure on a set $X$. A set $A \subset X$ is *$\mu$-measurable* if, for every $E \subset X$,

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A). \tag{1.1}$$

*Remark* 1.3. (1) Since a measure is countably sub-additive, it is sufficient to check the $\ge$ inequality in (1.1).
(2) In particular, it suffices to check (1.1) for $E \subset X$ with $\mu(E) < \infty$.
(3) If $A$ is $\mu$-measurable then so is $X \setminus A$.
(4) If $\mu(A) = 0$ then $A$ is $\mu$-measurable.

**Theorem 1.4.** *Let $\mu$ be a measure on a set $X$ and let $\mathcal{M}$ be the set of $\mu$-measurable subsets of $X$.*

(1) *If $A_1, A_2, \ldots \in \mathcal{M}$ then $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{M}$ and $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{M}$.*
(2) *$\mu$ is* countably additive *on $\mathcal{M}$. That is, if $A_1, A_2 \ldots \in \mathcal{M}$ are disjoint then*

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

(3) *If $A_1 \subset A_2 \subset \ldots \in \mathcal{M}$ then*

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{n\to\infty} \mu(A_i).$$

(4) *If $A_1 \supset A_2 \supset \ldots \in \mathcal{M}$ and $\mu(A_1) < \infty$ then*

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{n\to\infty} \mu(A_i).$$

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*Proof.* We first prove (1) for finite unions and intersections. If $A, B \in \mathcal{M}$ then for every $E \subset X$,

$$\begin{aligned}\mu(E) &= \mu(E \cap A) + \mu(E \setminus A)\\ &= \mu(E \cap A) + \mu((E \setminus A) \cap B) + \mu(E \setminus (A \cup B))\\ &\geq \mu(F \cap (A \cup B)) + \mu(E \setminus (A \cup B))\end{aligned}$$

by sub-additivity. Thus $A \cup B$ is $\mu$-measurable and induction gives finite unions. Taking complements gives finite intersections.

To prove (2) note that the inequality $\leq$ is given by sub-additivity. For the other inequality, for each $i \in \mathbb{N}$ let $A_i \in \mathcal{M}$ be disjoint and for each $j \in \mathcal{M}$ let

$$B_j = \bigcup_{i=1}^{j} A_i,$$

which is measurable by (1). Note that

$$B_j = B_{j-1} \cup A_j$$

and that this union is disjoint. Therefore, since $A_j$ is $\mu$-measurable,

$$\begin{aligned}\mu(B_j) &= \mu(B_j \cap A_j) + \mu(B_j \setminus A_j)\\ &= \mu(A_j) + \mu(B_{j-1}),\end{aligned}$$

since the $A_i$ are all disjoint. Therefore, by induction, $\mu(B_j) = \sum_{i=1}^{j} \mu(A_i)$ for each $j \in \mathbb{N}$. Finally, for each $j \in \mathbb{N}$, since $\mu$ is monotonic,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \geq \mu(B_j) = \sum_{i=1}^{j} \mu(A_i)$$

and so letting $j \to \infty$ gives (2).

(3) follows by applying (2) to the disjoint measurable sets $B_j = A_j \setminus A_{j-1}$.

(4) follows from (3) by setting $B_j = A_1 \setminus A_j$, so that

$$A_1 = \bigcap_{i\in\mathbb{N}} A_i \cup \bigcup_{i\in\mathbb{N}} B_i$$

and the $B_j$ increase. By sub-additivity,

$$\begin{aligned}\mu(A_1) &\leq \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) + \lim_{j\to\infty} \mu(B_j)\\ &= \mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) + \lim_{j\to\infty} \mu(A_1) - \mu(A_j),\end{aligned}$$

by applying (1) for finite unions. Since $\mu(A_1) < \infty$, (4) follows.

Finally, to prove (1) for countable unions, for each $j \in \mathbb{N}$ let

$$B_j = \bigcup_{i=1}^{j} A_i,$$

an increasing sequence, and let $E \subset X$ with $\mu(E) < \infty$. Define a function $\mu|_E$ on the power set of $X$ by $\mu|_E(A) = \mu(E \cap A)$. As shown in Exercise 1.7, $\mu|_E$ is a measure and any $\mu$-measurable set is $\mu|_E$-measurable. Therefore

$$\begin{aligned}\mu\left(E\cap\bigcup_{i\in\mathbb{N}}A_i\right)+\mu\left(E\setminus\bigcup_{i\in\mathbb{N}}A_i\right)&=\mu\left(E\cap\bigcup_{i\in\mathbb{N}}B_i\right)+\mu\left(E\setminus\bigcup_{i\in\mathbb{N}}B_i\right)\\&=\lim_{j\to\infty}\mu(E\cap B_j)+\lim_{j\to\infty}\mu(E\setminus B_j)\\&=\mu(E).\end{aligned}$$

Taking complements shows that countable intersections of measurable sets are measurable. ☐

**Definition 1.5.** A collection $\Sigma$ of subsets of a set $X$ is a *$\sigma$-algebra* if

(1) $\emptyset\in\Sigma$;
(2) $A\in\Sigma\Rightarrow X\setminus A\in\Sigma$;
(3) $A_1,A_2,\ldots\in\Sigma\Rightarrow\bigcup_{i\in\mathbb{N}}A_i\in\Sigma$.

Theorem 1.4 shows that the set of $\mu$-measurable sets is a $\sigma$-algebra.

For $\Omega$ a set of subset of a set $X$, the *$\sigma$-algebra generated by* $\Omega$ is

$$\Sigma(\Omega):=\bigcap\{\Sigma':\Sigma'\supset\Omega,\ \Sigma'\text{ a }\sigma\text{-algebra}\}.$$

By Exercise 1.2, it is a $\sigma$-algebra.

The *Borel $\sigma$-algebra* of a topological space $X$ is the $\sigma$-algebra generated by the open (respectively closed) subsets of $X$. Its elements are *Borel* subsets of $X$.

A measure for which all Borel sets are measurable is a *Borel measure*. It is *Borel regular* if for every $A\subset X$ there exists a Borel $B\supset A$ with $\mu(B)=\mu(A)$.

**Theorem 1.6** (Carathéodory criterion)**.** *Let $(X,d)$ be a metric space and $\mu$ a measure on $X$ which is additive on separated sets. That is, whenever $A,B\subset X$ with*

$$\inf\{d(x,y):x\in A,\ y\in B\}>0,$$

*we have*

$$\mu(A\cup B)=\mu(A)+\mu(B).$$

*Then $\mu$ is a Borel measure.*

*Proof.* Let $C\subset X$ be closed and $E\subset X$ with $\mu(E)<\infty$. We need to show

$$\mu(E)\geq\mu(E\cap C)+\mu(E\setminus C).$$

For each $j\in\mathbb{N}$ let

$$E_j=\{x\in E:\frac{1}{j+1}<\operatorname{dist}(x,C)\leq\frac{1}{j}\}$$

and

$$E_0=\{x\in E:\operatorname{dist}(x,C)>1\}.$$

Since $C$ is closed,

$$E\setminus C=E_0\cup\bigcup_{j\in\mathbb{N}}E_j.$$

Moreover, the $E_j$ with $j$ odd are pairwise separated so

$$\mu(E)\geq\mu(\bigcup_{j\text{ odd}}E_j)=\sum_{j\text{ odd}}\mu(E_j)$$

and so the sum is convergent. Similarly the sum over even indices is convergent and so

$$\sum_{j\geq n}\mu(E_j)\to 0\quad\text{as }n\to\infty.$$

Therefore

$$\begin{aligned}\mu(E) &\geq \mu\left(E \cap C \cup \bigcup_{j=0}^{n} E_j\right)\\ &= \mu(E \cap C) + \mu\left(\bigcup_{j=0}^{n} E_j\right)\\ &\geq \mu(E \cap C) + \mu(E \setminus C) - \sum_{j>n} \mu(E_j)\\ &\to \mu(E \cap C) + \mu(E \setminus C).\end{aligned}$$

□

**Definition 1.7** (Carathéodory construction)**.** Let $(X, d)$ be a metric space, $F$ a set of subsets of $X$ and $\zeta\colon F \to [0, \infty]$. For each $\delta > 0$ and $A \subset X$ define

$$\psi_\delta(A) = \inf \sum_{S \in G} \zeta(S),$$

where the infimum is taken over all countable

$$G \subset \{S \in F : \operatorname{diam}(S)\} < \delta$$

such that

$$A \subset \bigcup_{S \in G} S.$$

Finally, define $\psi(A) = \sup_{\delta>0} \psi_\delta(A)$.

For any $\delta > 0$, $\psi_\delta$ is a measure, as is $\psi$. Theorem 1.6 shows that $\psi$ is a Borel measure on $X$. Indeed, if $\operatorname{dist}(A, B) > \delta$ then

$$\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B).$$

If $F$ consists only of Borel sets then $\psi$ is Borel regular.

*Remark* 1.8. The fact that $\psi_{\delta'} \leq \psi_\delta$ whenever $\delta' \geq \delta$ implies that

$$\psi(A) = \lim_{\delta \to 0} \psi_\delta(A).$$

**Definition 1.9.** We define some properties of a measure $\mu$ on a topological space $X$.

(1) $\mu$ is *locally finite* if every point in $X$ has a neighbourhood of finite measure.
(2) $\mu$ is *$\sigma$-finite* if there exist measurable $X_i \subset X$ with $\mu(X_i) < \infty$ and $X = \bigcup_{i \in \mathbb{N}} X_i$.
(3) $\mu$ is *finite* if $\mu(X) < \infty$.
(4) A Borel measure $\mu$ is a *Radon* measure if
  (a) $\mu(K) < \infty$ for all compact $K \subset X$,
  (b) $\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\}$ for all Borel $A \subset X$.
  (c) $\mu(A) = \inf\{\mu(U) : U \supset A \text{ open}\}$ for all Borel $A \subset X$.

**Definition 1.10.** Let $\mu$ be a measure on a set $X$. A property of points in $X$ holds *$\mu$ almost everywhere* (or $\mu$-a.e.) if the set of points for which the property doesn't hold has $\mu$ measure zero.

## 1.1. **Exercises.**

**Exercise 1.1.** Usually in measure theory, a measure is defined as a countably additive function defined on a $\sigma$-algebra. However, using our definition is simply a convenience rather than a restriction.

Indeed, suppose $\mu$ is a countably additive function defined on a $\sigma$-algebra $\Sigma$ of $X$. Show that it can be extended to the power set of $X$ by

$$\mu(A) = \inf\{\mu(B) : A \subset B \in \Sigma\}$$

and that any $B \in \Sigma$ is $\mu$-measurable.

Conversely, any measure is countably additive when restricted to any $\sigma$-algebra of measurable sets.

**Exercise 1.2.** Let $\Omega$ be a set of subsets of a set $X$. Show that $\Sigma(\Omega)$ is a $\sigma$-algebra. Note that it is the smallest $\sigma$-algebra of $X$ containing $\Omega$.

**Exercise 1.3.** Show that the following sets are Borel subsets of $\mathbb{R}$: $\mathbb{Q}$, $[0, 1)$, the set of points in $[0, 1]$ whose first decimal is even.

Let $f : [0, 1] \to [0, 1]$. Show that the set of points where $f$ is continuous is a Borel set. What about the set of points where $f$ is differentiable?

**Exercise 1.4.** Let $X$ be a set and $x \in X$. The *Dirac measure at* $x$ is defined as $\delta_x(A) = 1$ if $x \in A$, $\delta_x(A) = 0$ otherwise. Show that $\delta_x$ is a measure on $X$. What are its measurable sets?

Define the *counting measure* on $X$ to be the cardinality (finite or $\infty$) of any subset of $X$. Show that this is a measure. What are its measurable sets?

**Exercise 1.5.** The *Lebesgue measure* on $\mathbb{R}^n$, denoted $\mathcal{L}^n$, is defined using the Carathéodory construction with $F$ the set of cubes and $\zeta(Q) = \mathrm{vol}(Q)$. Its measurable sets are called the *Lebesgue measurable* subsets.

For $(X, d)$ a metric space and $s \geq 0$ the *$s$-dimensional Hausdorff measure* on $X$, denoted $\mathcal{H}^s$, is defined using the Carathéodory construction with $F$ the set of all sets and $\zeta(S) = \mathrm{diam}(S)^s$.

(1) Show that $\mathcal{L}^n$ is a non-zero, translation invariant and $n$-homogenous measure. That is, for any $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $t > 0$, $\mathcal{L}^n(A + x) = \mathcal{L}^n(A)$ and $\mathcal{L}^n(tA) = t^n\mathcal{L}^n(A)$.
(2) On $\mathbb{R}^n$ show that $\mathcal{H}^n = c\mathcal{L}^n$ for some constant $c > 0$. What is $c$?
(3) Let $f : X \to Y$ be an $L$-Lipschitz function between two metric spaces. Show that for any $s \geq 0$ and $A \subset X$,
$$\mathcal{H}^s(f(A)) \leq L^s\mathcal{H}^s(A).$$
(4) For any metric space $X$, show that $\mathcal{H}^0$ is the counting measure on $X$.
(5) For $0 \leq s < t < \infty$, suppose that $\mathcal{H}^s(A) < \infty$. Show that $\mathcal{H}^t(A) = 0$. Hence there exists a single $0 \leq s \leq \infty$ for which $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = \infty$ for all $t < s$. This $t$ is called the *Hausdorff dimension* of $A$, denoted $\dim_{\mathrm{H}} A$.

**Exercise 1.6.** The *Cantor set* $K \subset [0, 1]$ is defined as follows. Let $K_0 = [0, 1]$ and for each $i \in \mathbb{N}$ let $K_i$ be obtained from deleting the "middle third" open interval from each of the intervals in $K_{i-1}$. That is, $K_1 = [0, 1/3] \cup [2/3, 1]$,

$$K_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1],$$

etc. Define $K = \bigcap_{i\in\mathbb{N}} K_i$. Note that $K$ is compact and hence Borel.

(1) Show that $K$ is uncountable.
(2) Let $s = \log 2/\log 3$. Show that $0 < \mathcal{H}^s(K) < \infty$.

In particular, $K$ is an uncountable subset of $\mathbb{R}$ with $\mathcal{L}^1(K) = 0$.

**Exercise 1.7.** If $\mu$ is a measure on a set $X$ and $S \subset X$, the *restriction* of $\mu$ to $A$ is defined as

$$\mu|_S(A) := \mu(S \cap A).$$

Show that $\mu|_S$ is a measure on $X$ and any $\mu$-measurable set is also $\mu|_S$-measurable (even if $S$ is not $\mu$-measurable).

Give an examples of $S \subset \mathbb{R}^2$ with $\dim_{\mathrm{H}} S = 1$ for which $\mathcal{H}^1|_S$ is:

(1) finite,
(2) $\sigma$-finite but not finite,
(3) not $\sigma$-finite.

**Exercise 1.8.** The fundamental properties of measures are those given in Theorem 1.4, in particular countable additivity. It is necessary for us to only require this to be true for measurable sets, as can be seen from the existence of non-measurable sets.

Define the *Vitali set* as follows. Consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ iff $x - y \in \mathbb{Q}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, each equivalence class $V_x$ intersects $[0, 1]$. Therefore, by the axiom of choice(!), we may construct a set $\mathcal{V} \subset [0, 1]$ consisting of exactly one member of each equivalence class.

Show:

(1) If $p \neq q$ are rational then $p + \mathcal{V}$ and $q + \mathcal{V}$ are disjoint.
(2) $[0, 1] \subset \bigcup\{q + \mathcal{V} : q \in \mathbb{Q} \cap [-1, 1]\} \subset [-1, 2]$.
(3) Show that $\mathcal{L}^1(\mathcal{V}) \neq 0$.
(4) Deduce that $\mathcal{V}$ is not Lebesgue measurable.

**Exercise 1.9.** Let $\mu$ be a finite measure on a metric space $X$. Prove that for every Borel $B \subset X$,

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ closed}\} \tag{1.2}$$

and

$$\mu(B) = \inf\{\mu(U) : U \supset B \text{ open}\}. \tag{1.3}$$

Property (1.2) is called *inner regularity by closed sets* and (1.3) is called *outer regularity by open sets*.

Hint: observe that it suffices to show that all Borel sets satisfy (1.2). Show that the set

$$\{B \subset X : B \text{ and } X \setminus B \text{ satisfy (1.2)}\}$$

is a $\sigma$-algebra that contains all closed subsets of $X$.

Show that this result is true if $\mu$ is assumed to only be $\sigma$-finite.

**Exercise 1.10.** Let $X$ be a complete and separable metric space. Show that any finite Borel measure on $X$ is a Radon measure.

Hint: a metric space is compact if and only if it is complete and totally bounded.

## 2. Integration

**Definition 2.1.** Let $\mu$ be a measure on a set $X$. A function $f\colon X \to \mathbb{R}$ is *$\mu$-measurable* if $f^{-1}((a,\infty))$ is $\mu$-measurable for every $a \in \mathbb{R}$.

Let $X$ be a topological space. A function $f\colon X \to \mathbb{R}$ is a *Borel function* if $f^{-1}((a,\infty))$ is a Borel set for every $a \in \mathbb{R}$.

**Definition 2.2.** Let $\mu$ be a measure on a set $X$. A *simple function* is any function of the form

$$\sum_{i=1}^{n} a_i \chi_{A_i},$$

where $n \in \mathbb{N}$, each $a_i \geq 0$ and each $A_i \subset X$ is $\mu$-measurable.

The *integral with respect to $\mu$* of a simple function is

$$\int_X \sum_{i=1}^{n} a_i \chi_{A_i} \,\mathrm{d}\mu := \sum_{i=1}^{n} a_i \mu(A_i),$$

where we treat $0 \cdot \infty = 0$.

**Definition 2.3.** Let $\mu$ be a measure on a set $X$ and let $f\colon X \to \mathbb{R}^+$ be $\mu$-measurable. We define the *integral of $f$ with respect to $\mu$* to be

$$\int_X f \,\mathrm{d}\mu := \sup \left\{ \int_X s \,\mathrm{d}\mu : 0 \leq s \leq f,\ s \text{ simple} \right\}.$$

This agrees with the previous definition when $f$ is simple.

If $f\colon X \to \mathbb{R}$ is measurable, let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ (both measurable functions), so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. If $\int_X f^+ \,\mathrm{d}\mu < \infty$ and $\int_X f^- \,\mathrm{d}\mu < \infty$ then we say that $f$ is *$\mu$-integrable* and we define the *integral of $f$ with respect to $\mu$* to be

$$\int_X f \,\mathrm{d}\mu = \int_X f^+ \,\mathrm{d}\mu - \int_X f^- \,\mathrm{d}\mu.$$

**Theorem 2.4** (Monotone convergence theorem)**.** *Let $\mu$ be a measure on a set $X$ and $f_k\colon X \to [0,\infty]$ $\mu$-measurable. Suppose that for every $x \in X$, $f_{n+1}(x) \geq f_n(x)$ for all $n \in \mathbb{N}$ and $f_n(x) \to f(x)$. Then*

$$\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu.$$

*Proof.* First, $f$ is $\mu$-measurable by Exercise 2.2. We prove the theorem under the additional assumption that $\int f \,\mathrm{d}\mu < \infty$. For the other case, see Exercise 2.4.

Let $0 \leq s \leq f$ be a simple function,

$$s = \sum_{k=1}^{m} a_k \chi_{A_k}.$$

Since

$$\int s \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu,$$

$\mu(A_k) < \infty$ for each $1eqk \leq m$.

Fix $\epsilon > 0$ and for each $1 \leq k \leq m$ let

$$G_n^k = \{x \in A_k : f_n(x) \geq (1-\epsilon) f(x) \geq (1 - \epsilon a_k)\}.$$

Then since the $f_n$ monotonically increase, $G_n^k$ increase to $A_k$ as $n \to \infty$. Therefore there exists $N \in \mathbb{N}$ such that $\mu(A_k \setminus G_n^k) < \epsilon$ for all $n \geq N$ and all $1 \leq k \leq m$. Then for all $n \geq N$,

$$\begin{aligned}
\int f_n \,\mathrm{d}\mu &\geq \sum_{k=1}^{m} \int_{G_n^k} f_n \,\mathrm{d}\mu \\
&= \sum_{k=1}^{m} \int_{G_n^k} (1-\epsilon)A_k \\
&= \sum_{k=1}^{m} \int_{X} (1-\epsilon)a_k\chi_{A_k} - (1-\epsilon)a_k\mu(A_k \setminus G_n^k) \\
&\geq (1-\epsilon)\int_X s\,\mathrm{d}\mu - \epsilon\sum_{k=1}^{m} a_k.
\end{aligned}$$

The final term is a fixed (depending only upon $s$) multiple of $\epsilon$. That is, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\int_X f_n \,\mathrm{d}\mu \geq \int_X s\,\mathrm{d}\mu - \epsilon.$$

This is true for all simple $0 \leq s \leq f$ and so

$$\liminf \int_X f_n \,\mathrm{d}\mu \geq \int_X f\,\mathrm{d}\mu.$$

The other inequality is true by the monotonicity of the integral. □

**Theorem 2.5** (Fatou's lemma). *Let $\mu$ be a measure on a set $X$ and $f_k\colon X \to [0,\infty]$ measurable. Then*

$$\int_X \liminf f_k \,\mathrm{d}\mu \leq \liminf \int_X f_k\,\mathrm{d}\mu.$$

*Proof.* Apply the monotone convergence theorem to

$$g_k(x) := \inf_{n\geq k} f_n(x)$$

to get

$$\int_X \liminf f_k \,\mathrm{d}\mu = \lim \int_X g_k\,\mathrm{d}\mu \leq \liminf \int_X f_k\,\mathrm{d}\mu.$$

□

*Remark* 2.6 (Reverse Fatou). Suppose that there exists $g \geq 0$ with $\int g\,\mathrm{d}\mu < \infty$ and $f_k \leq g$ for all $k$. Then

$$\limsup_X f_k \,\mathrm{d}\mu \geq \limsup \int_X f_k\,\mathrm{d}\mu.$$

Indeed, this follows by applying Fatou's lemma to $g - f_k$.

**Theorem 2.7.** *Let $\mu$ be a measure on $X$ and $f_n\colon X \to \mathbb{R}$ measurable such that $f_n \to f$ pointwise. Suppose that there exists measurable $g\colon X \to [0,\infty]$ with $\int g\,\mathrm{d}\mu < \infty$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Then $f$ is integrable and*

$$\int f_n\,\mathrm{d}\mu \to \int f\,\mathrm{d}\mu.$$

*Proof.* Observe that for all $n \in \mathbb{N}$, $|f - f_n| \leq 2g$ and that $\limsup |f - f_n| = 0$. Then by the reverse Fatou lemma,

$$\left| \int f \, \mathrm{d}\mu - \int f_n \, \mathrm{d}\mu \right| \leq \int |f - f_n| \, \mathrm{d}\mu \to 0.$$

□

2.1. **Exercises.**

**Exercise 2.1.** For $\mu$ a measure on a set $X$, let $f \colon X \to \mathbb{R}$ be measurable, respectively Borel. Show that the pre-image of any Borel $B \subset \mathbb{R}$ is $\mu$-measurable, respectively Borel. Compare this to the definition of a continuous function.

**Exercise 2.2.** Let $\mu$ be a measure on $X$ and for each $i \in \mathbb{N}$ let $f_i \colon X \to \mathbb{R}$ be $\mu$-measurable. Show that the functions

$$\limsup_{i\to\infty} f_i \quad \text{and} \quad \liminf_{i\to\infty} f_i$$

are $\mu$-measurable.

Show that a linear combination of $\mu$-measurable functions is $\mu$-measurable. Show that a countable (pointwise) sum of $\mu$-measurable functions is $\mu$-measurable.

**Exercise 2.3.** There are some simple properties of the integral to check:

(1) If $f \leq g$ $\mu$-a.e. then
$$\int f d\mu \leq \int g \, \mathrm{d}\mu;$$
(2) The integral with respect to $\mu$ is a linear operator;
(3) If $S \subset X$ is $\mu$-measurable then
$$\int_X f \, \mathrm{d}\mu = \int_S f \, \mathrm{d}\mu + \int_{X \setminus S} f \, \mathrm{d}\mu;$$
(4) $|\int f \, \mathrm{d}\mu| \leq \int |f| \, \mathrm{d}\mu$;
(5) etc...

**Exercise 2.4.** Prove the monotone convergence theorem for the case $\int f \, \mathrm{d}\mu = \infty$.

Prove that if a sequence of measurable functions $f_n$ monotonically decrease to $f$ and $\int f_1 \, \mathrm{d}\mu < \infty$ then $\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$.

**Exercise 2.5.** Show that the monotone convergence theorem is false if the sequence does not monotonically increase or are not uniformly bounded below.

## 3. Some standard theorems

**Theorem 3.1.** *Let $\mu, \nu$ be finite Borel measures on a set $X$. There exists a Borel function $h \colon X \to [0, \infty)$ and a Borel set $A \subset X$ such that $\nu(A) = 0$ and*

$$\mu(B) = \int_B h \, \mathrm{d}\nu + \mu(A \cap B) \quad \textit{for all Borel } B \subset X.$$

*The measure $\mu|_{X \setminus A}$ is called the* absolutely continuous *part, and $\mu|_A$ the* singular part*, of $\mu$ with respect to $\nu$. If we assume $\mu \ll \nu$, that is $\nu(S) = 0 \implies \mu(S) = 0$ for any $S \subset X$, then we have $\mu(A) = 0$ and we call $h$ the* Radon-Nikodym derivative *of $\mu$ with respect to $\nu$.*

*Proof.* Define a continuous linear function on the space $L^2(\mu+\nu)$ by

$$g \mapsto \int_X g \,\mathrm{d}\mu.$$

By the Riesz representation theorem (see Exercise 3.1), there exists a Borel $f \in L^2(\mu+\nu)$ such that

$$\int_X g \,\mathrm{d}\mu = \int gf \,\mathrm{d}(\mu+\nu). \tag{3.1}$$

Set $A = \{x \in X : f(x) = 1\}$, a Borel set. Note that (3.1) implies $\nu(A) = 0$. For any Borel $B \subset X \setminus A$ and $n \in \mathbb{N}$, let

$$S_n = \{x \in B : |1 - f(x)| > 1/n\}.$$

Then

$$g = \frac{\chi_{S_n}}{1-f}$$

is bounded and hence in $\in L^2(\mu+\nu)$ and (3.1) implies

$$\mu(S_n) = \int_{S_n} \frac{1-f}{1-f} \,\mathrm{d}\mu = \int_{S_n} \frac{f}{1-f} \,\mathrm{d}\nu.$$

Therefore, by the monotone convergence theorem,

$$\mu(B) = \int_B \frac{f}{1-f} \,\mathrm{d}\nu.$$

Setting $h = f/(1-f)$ on $X \setminus A$ and $h = 0$ on $A$ completes the proof. □

**Theorem 3.2** (Egorov's theorem)**.** *Let $\mu$ be a finite measure on a set $X$ and $f_n \colon X \to \mathbb{R}$ a sequence of $\mu$-measurable functions such that $f_n \to f$ pointwise $\mu$-a.e. Then for every $\epsilon > 0$ there exists a measurable $G \subset X$ with $\mu(X \setminus G) < \epsilon$ and $f_n \to f$ uniformly on $G$.*

*Proof.* Fix $\epsilon > 0$, $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$ let

$$B_{n,k} = \{x \in X : |f_n(x) - f(x)| > 1/k \text{ for some } m \geq n\}.$$

By assumption, the $B_{n,k}$ are measurable sets that monotonically decrease to a $\mu$-null set as $n \to \infty$. Therefore, there exists $n \in \mathbb{N}$ such that $\mu(B_{n,k}) < \epsilon 2^{-k}$. Let

$$G_k = X \setminus B_{n,k} \quad \text{and} \quad G = \bigcap_{k\in\mathbb{N}} G_k.$$

Then $\mu(X \setminus G) < \epsilon$ and, for each $x \in G$ and each $k \in \mathbb{N}$, $G \subset G_k$, so there exists $n \in \mathbb{N}$ such that

$$|f(x) - f_m(x)| < 1/k$$

for all $m \geq n$. That is, $f_m \to f$ uniformly on $G$, as required. □

**Theorem 3.3.** *Let $\mu$ be a finite Radon measure on a topological space $X$ and let $f \colon X \to \mathbb{R}$ be $\mu$-measurable. Then for every $\epsilon > 0$ there exists a compact $K \subset X$ with $\mu(X \setminus K) < \epsilon$ such that $f|_K$ is continuous.*

*Proof.* Fix $\epsilon > 0$ and for each $i \in \mathbb{Z}$ let

$$X_i = f^{-1}([i\epsilon, (i+1)\epsilon)),$$

a collection of disjoint Borel sets which cover $X$. Since $\mu$ is Radon, for every $i \in \mathbb{N}$ there exists a compact $K_i \subset X_i$ with $\mu(X_i \setminus K_i) < \epsilon 2^{-i}$. Since $\mu(X) < \infty$, there exists $n \in \mathbb{N}$ such that

$$\mu\left(X \setminus \bigcup_{i=1}^{n} K_i\right) < \epsilon.$$

Let $K_\epsilon = \bigcup_{i=1}^n K_i$, a compact set, and $f_\epsilon = \sum_{i=1}^n t_i \chi_{K_i}$. Since the $K_i$ are disjoint and compact, $f_\epsilon$ is continuous on $K_\epsilon$. Further, $\|f - f_\epsilon\|_\infty < \epsilon$ on $K_\epsilon$. Repeat this for each $k \in \mathbb{N}$ for $\epsilon_k = 2^{-k}\epsilon$ and let $K = \bigcap_{k\in\mathbb{N}} K_{\epsilon_k}$, so that $\mu(X \setminus K) < \epsilon$. Then $f$ is the uniform limit of the $f_{\epsilon k}$ on $K$, and so $f$ is continuous on $K$. □

**Definition 3.4.** Let $X$ and $Y$ be sets and $\Sigma_X$ and $\Sigma_Y$ two $\sigma$-algebras on $X$ and $Y$ respectively. The set of *rectangles* generated by $\Sigma_X$ and $\Sigma_Y$ is

$$\Sigma_X \otimes \Sigma_Y := \{A \times B : A \in \Sigma_X,\ B \in \Sigma_Y\}.$$

It is a $\sigma$-algebra on $X \times Y$.

Let $\mu, \nu$ be measures on $X, Y$ respectively and let

$$\mathcal{R} = \{A \subset X : A\ \mu\text{-measurable}\} \otimes \{B \subset Y : B\ \nu\text{-measurable}\}.$$

The *product measure* $\mu \times \nu$ on $X \times Y$ is defined by

$$\mu \times \nu(S) = \inf \sum_{i\in\mathbb{N}} \mu(A_i)\nu(B_i),$$

where the infimum is taken over all countable collections of rectangles $A_i \times B_i \in \mathcal{R}$ with

$$S \subset \bigcup_{i\in\mathbb{N}} A_i \times B_i.$$

This is a measure, see Exercise 3.2.

**Lemma 3.5.** *Let $X, Y$ be sets and $\mu, \nu$ measures on $X, Y$ respectively. Then $\mu \times \nu$ is equivalently defined by the formula*

$$\mu \times \nu(S) = \inf \sum_{i\in\mathbb{N}} \mu(A_i)\nu(B_i),$$

*where the infimum is taken over all* disjoint *countable collections of rectangles $A_i \times B_i \in \mathcal{R}$ with*

$$S \subset \bigcup_{i\in\mathbb{N}} A_i \times B_i.$$

*Proof.* If $A, C$ are $\mu$-measurable and $B, D$ are $\nu$-measurable then

$$\begin{aligned}(A \times B) \setminus (C \times D) &= [(A \setminus C) \times B] \cup [(A \cap C) \times (B \setminus D)] \\ &:= A_1 \times B_1 \cup A_2 \times B_2\end{aligned}$$

is a decomposition into disjoint rectangles. Since $A, C$ and $B, D$ are $\mu$ and $\nu$ measurable respectively,

$$\begin{aligned}\mu(A_1)\nu(B_1) + \mu(A_2)\nu(B_2) &= [\mu(A) - \mu(A \cap C)]\mu(B) + \mu(A \cap C)[\nu(B) - \nu(B \cap D)] \\ &= \mu(A)\nu(B) - \mu(A \cap C)\nu(B \cap D) \\ &\leq \mu(A)\nu(B).\end{aligned}$$

Thus the two formulae agree. □

**Theorem 3.6** (Fubini's theorem)**.** *Let $X, Y$ be sets and $\mu, \nu$ measures on $X, Y$ respectively.*

(1) *If $A$ is $\mu$-measurable and $B$ is $\nu$-measurable then $A \times B$ is $\mu \times \nu$-measurable and*
$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$
(2) *If $S$ is $\mu \times \nu$-measurable then*
$$S^y := \{x \in X : (x, y) \in S\}$$
*is $\mu$-measurable for $\nu$-a.e. $y \in Y$ and*
$$S_x := \{y \in Y : (x, y) \in S\}$$
*is $\nu$-measurable for $\mu$-a.e. $x \in X$ and*
$$\mu \times \nu(S) = \int_Y \mu(S^y)\,\mathrm{d}\nu(y) = \int_X \nu(S_x)\,\mathrm{d}\mu(x).$$
(3) *If $f\colon X \times Y \to \mathbb{R}^+$ is $\mu \times \nu$-measurable or $f\colon X \times Y \to \mathbb{R}$ is $\mu \times \nu$ integrable then*
$$\int_{X \times Y} f\,\mathrm{d}\mu \times \nu = \int_X \int_Y f\,\mathrm{d}\nu(y)\,\mathrm{d}\mu(x) = \int_Y \int_X f\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

*Proof.* Note that (3) follows from (2) by the monotone convergence theorem.

To begin, let
$$\mathcal{U} := \left\{ \bigcup_{i \in \mathbb{N}} R_i : R_i \in \mathcal{R} \text{ pairwise disjoint} \right\}$$
and, for $S \subset X \times Y$, define $\sigma_S\colon Y \to \mathbb{R}$ by
$$\sigma_S(y) = \mu(S^y).$$
Further, let
$$\mathcal{P} := \{S \subset X \times Y : \sigma_S \text{ is } \nu\text{-measurable}\}$$
and for any $S \in \mathcal{P}$ define
$$\rho(S) := \int_Y \sigma_S\,\mathrm{d}\nu.$$
Observe that $\sigma_S$ and hence $\rho(S)$ are monotonic in $S$.

If $S = A \times B \in \mathcal{R}$ then $\sigma_S = \mu(A)\chi_B$ is $\nu$-measurable and $\rho(S) = \mu(A)\nu(B)$. If $U \in \mathcal{U}$ with
$$U = \bigcup_{i \in \mathbb{N}} A_i \times B_i$$
a disjoint union, then
$$\sigma_U = \sum_{i \in \mathbb{N}} \mu(A_i)\chi_{B_i}$$
is a countable sum of $\nu$-measurable functions and so $U \in \mathcal{P}$. Moreover,
$$\rho(U) = \int_Y \sigma_U\,\mathrm{d}\nu = \sum_{i \in \mathbb{N}} \mu(A_i)\nu(B_i). \tag{3.2}$$
Thus, for any $S \subset X \times Y$,
$$\mu \times \nu(S) = \inf\{\rho(U) : S \subset U \in \mathcal{U}\}. \tag{3.3}$$

To prove (1), let $A$ be $\mu$-measurable and $B$ be $\nu$-measurable. By definition, $\rho(A \times B) = \mu(A)\nu(B)$ and, since $\rho$ is monotonic, $\rho(A \times B) \le \rho(U)$ whenever $A \times B \subset U \in \mathcal{U}$. Thus, by (3.3),
$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$

Let $E \subset X \times Y$ and $E \subset U \in \mathcal{U}$. Observe

$$U \cap (A \times B) \quad \text{and} \quad U \setminus (A \times B)$$

are disjoint members of $\mathcal{U}$. Therefore, by (3.2) and (3.3),

$$\begin{aligned} \rho(U) &= \rho(U \cap (A \times B)) + \rho(V \setminus (A \times B)) \\ &\geq \mu \times \nu(E \cap (A \times B)) + \mu \times \nu(E \setminus (A \times B)). \end{aligned}$$

When taking the infimum over all such $U$, the left hand side converges to $\mu \times \nu(E)$, and so $A \times B$ is $\mu \times \nu$-measurable. This also implies that all elements of $\mathcal{U}$ are measurable.

Let $S \subset X \times Y$ and suppose that $U_1, U_2, \ldots \in \mathcal{U}$ are such that $\rho(U_i) \to \mu \times \nu(S)$. Since the intersection of any two rectangles is a rectangle,

$$V_i := U_i \cap U_{i-1} \cap \ldots U_1 \in \mathcal{U}$$

for each $i \in \mathbb{N}$. Moreover, $\rho(V_i)$ monotonically decreases to $\mu \times \nu(S)$. Let $W = \bigcap_{i \in \mathbb{N}} V_i \supset S$. Note that $\sigma_{V_i}$ monotonically decreases to $\sigma_W$, so that $W \in \mathcal{P}$. Since $\mu, \nu$ are finite, the monotone convergence theorem implies that $\rho(V_i) \to \rho(W)$ and hence $\mu \times \nu(S) = \rho(W)$. Since $\mu \times \nu(W) \geq \mu \times \nu(S)$, the monotonicity of $\rho$ implies $\mu \times \nu(W) = \rho(W)$.

To prove (2), if $S$ is $\mu \times \nu$-measurable, then $\mu \times \nu(W \setminus S) = 0$. By (3.3) there exists $Z \supset W \setminus U$ with $\rho(Z) = 0$. That is, $\mu(W^y) = \mu(S^y)$ for $\nu$-a.e. $y \in Y$, and hence the first conclusion of (2). Moreover, $\rho(S) = \rho(W) = \mu \times \nu(S)$, which concludes the proof. □

### 3.1. **Exercises.**

**Exercise 3.1.**

**Exercise 3.2.** Let $X, Y$ be sets and $\mu, \nu$ measures on $X, Y$ respectively. Prove that $\mu \times \nu$ is a measure on $X \times Y$.

**Exercise 3.3.** Let $X, Y$ be separable metric spaces. Show that

$$\mathcal{B}(X \times Y) = \Sigma(\mathcal{B}(X) \otimes \mathcal{B}(Y)).$$

**Exercise 3.4.** Let $X, Y$ be separable metric spaces and $\mu, \nu$ finite Borel measures on $X, Y$ respectively. Show that $\mu \times \nu|_{\mathcal{B}(X \times Y)}$ is the unique countably additive set function on $\mathcal{B}(X \times Y)$ satisfying $\mu(A \times B) = \mu(A) \times \nu(B)$ for all $A \times B \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$.

**Exercise 3.5.** Prove Fubini's theorem and Exercises 3.3 and 3.4 for $\sigma$-finite measures $\mu, \nu$.

## Part 2. Some topics in Geometric Measure Theory

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