
Graduate real analysis

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MEASURE THEORY

| | | |
|-----------|--|-----------|
| 1 | Measures | 3 |
| 2 | Integration | 13 |
| 3 | The Daniell integral | 17 |
| 4 | Covering theorems | 25 |
| 5 | Differentiability of Lipschitz functions | 31 |
| 6 | Hausdorff measure and densities | 37 |
| 7 | Rectifiable sets and approximate tangent planes | 41 |
| 8 | Purely unrectifiable sets | 47 |
| 9 | Some standard theorems | 57 |
| 10 | Fubini's theorem | 63 |
| 11 | Index | 69 |
| | Proof Index | 71 |
| | Index | 75 |

These notes are also available as [pdf](#).

MEASURES

We wish to assign a value to the size of subsets of some given space, such as the length, area or volume of subsets of \mathbb{R}^m .

Definition 1 (Measure)

A *measure* μ on a set X is a function

$$\mu: \{A : A \subset X\} \rightarrow [0, \infty]$$

such that

1. $\mu(\emptyset) = 0$;
 2. $\mu(A) \subset \mu(B)$ whenever $A \subset B \subset X$ (μ is *monotonic*);
 3. $\mu(\bigcup_{i \in \mathbb{N}} A_i) \leq \sum_{i \in \mathbb{N}} \mu(A_i)$ whenever $A_1, A_2, \dots \subset X$ (μ is *countably subadditive*).
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Definition 2 (Measurable set)

Let μ be a measure on a set X . A set $A \subset X$ is μ -measurable if, for every $E \subset X$,

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A). \tag{1.1}$$

Remark 1

1. Since a measure is countably sub-additive, it is sufficient to check the \geq inequality in (1.1).
 2. In particular, it suffices to check (1.1) for $E \subset X$ with $\mu(E) < \infty$.
 3. If A is μ -measurable then so is $X \setminus A$.
 4. If $\mu(A) = 0$ then A is μ -measurable.
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Definition 3 (Restriction)

If μ is a measure on a set X and $S \subset X$, the *restriction* of μ to A is defined as

$$\mu|_S(A) := \mu(S \cap A).$$

Lemma 1

Let μ be a measure on a set X and $S \subset X$. Then $\mu|_S$ is a measure on X and any μ -measurable set is also $\mu|_S$ -measurable.

Proof. The fact that $\mu|_S$ is a measure follows immediately from the fact that μ is a measure. If $A \subset X$ is μ -measurable, then for any $E \subset X$,

$$\begin{aligned}\mu|_S(E) &= \mu(E \cap S) = \mu(E \cap S \cap A) + \mu(E \cap S \setminus A) \\ &= \mu|_S(E \cap A) + \mu|_S(E \setminus A),\end{aligned}$$

as required.

Theorem 1

Let μ be a measure on a set X and let \mathcal{M} be the set of μ -measurable subsets of X .

1. If $A_1, A_2, \dots \in \mathcal{M}$ then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{M}$ and $\bigcap_{i \in \mathbb{N}} A_i \in \mathcal{M}$.*
2. μ is countably additive on \mathcal{M} . That is, if $A_1, A_2, \dots \in \mathcal{M}$ are disjoint then

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i).$$

3. If $A_1 \subset A_2 \subset \dots \in \mathcal{M}$ then

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

4. If $A_1 \supset A_2 \supset \dots \in \mathcal{M}$ and $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{i \in \mathbb{N}} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

Proof. We first prove the first point for finite unions and intersections. If $A, B \in \mathcal{M}$ then for every $E \subset X$,

$$\begin{aligned}\mu(E) &= \mu(E \cap A) + \mu(E \setminus A) \\ &= \mu(E \cap A) + \mu((E \setminus A) \cap B) + \mu(E \setminus (A \cup B)) \\ &\geq \mu(E \cap (A \cup B)) + \mu(E \setminus (A \cup B))\end{aligned}$$

by sub-additivity. Thus $A \cup B$ is μ -measurable and induction gives finite unions. Taking complements gives finite intersections.

To prove the second point, note that the inequality \leq is given by sub-additivity. For the other inequality, for each $i \in \mathbb{N}$ let $A_i \in \mathcal{M}$ be disjoint and for each $j \in \mathbb{N}$ let

$$B_j = \bigcup_{i=1}^j A_i,$$

which is measurable by the first conclusion. Note that

$$B_j = B_{j-1} \cup A_j$$

and that this union is disjoint. Therefore, since A_j is μ -measurable,

$$\begin{aligned} \mu(B_j) &= \mu(B_j \cap A_j) + \mu(B_j \setminus A_j) \\ &= \mu(A_j) + \mu(B_{j-1}), \end{aligned}$$

since the A_i are all disjoint. Therefore, by induction, $\mu(B_j) = \sum_{i=1}^j \mu(A_i)$ for each $j \in \mathbb{N}$. Finally, for each $j \in \mathbb{N}$, since μ is monotonic,

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \geq \mu(B_j) = \sum_{i=1}^j \mu(A_i)$$

and so letting $j \rightarrow \infty$ gives the second conclusion.

The third conclusion follows by applying the second conclusion to the disjoint measurable sets $B_j = A_j \setminus A_{j-1}$.

The fourth conclusion follows from the third by setting $B_j = A_1 \setminus A_j$, so that

$$A_1 = \bigcap_{i \in \mathbb{N}} A_i \cup \bigcup_{i \in \mathbb{N}} B_i$$

and the B_j increase. By sub-additivity,

$$\begin{aligned} \mu(A_1) &\leq \mu\left(\bigcap_{i \in \mathbb{N}} A_i\right) + \lim_{j \rightarrow \infty} \mu(B_j) \\ &= \mu\left(\bigcap_{i \in \mathbb{N}} A_i\right) + \lim_{j \rightarrow \infty} (\mu(A_1) - \mu(A_j)), \end{aligned}$$

by applying the first conclusion for finite unions. Since $\mu(A_1) < \infty$, the fourth point follows.

Finally, to prove the first point for countable unions, for each $j \in \mathbb{N}$ let

$$B_j = \bigcup_{i=1}^j A_i,$$

an increasing sequence, and let $E \subset X$ with $\mu(E) < \infty$. Since the B_j are μ -measurable,

$$\begin{aligned} \mu(E) &= \lim_{j \rightarrow \infty} \mu(E \cap B_j) + \lim_{j \rightarrow \infty} \mu(E \setminus B_j) \\ &= \mu\left(E \cap \bigcup_{i \in \mathbb{N}} B_i\right) + \mu\left(E \setminus \bigcup_{i \in \mathbb{N}} B_i\right) \\ &= \mu\left(E \cap \bigcup_{i \in \mathbb{N}} A_i\right) + \mu\left(E \setminus \bigcup_{i \in \mathbb{N}} A_i\right), \end{aligned}$$

using the fact that the B_j are $\mu|_E$ -measurable in the second equality. Taking complements shows that countable intersections of measurable sets are measurable.

Definition 4 (Sigma algebra)

A collection Σ of subsets of a set X is a σ -algebra if

1. $\emptyset \in \Sigma$;
2. $A \in \Sigma \Rightarrow X \setminus A \in \Sigma$;
3. $A_1, A_2, \dots \in \Sigma \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \Sigma$.

Theorem 1 shows that the set of μ -measurable sets is a σ -algebra.

For Ω a set or subset of a set X , the σ -algebra generated by Ω is

$$\Sigma(\Omega) := \bigcap \{ \Sigma' : \Sigma' \supset \Omega, \Sigma' \text{ a } \sigma\text{-algebra} \}.$$

By *Example 2*, it is a σ -algebra.

The Borel σ -algebra* of a topological space X is the σ -algebra generated by the open (respectively closed) subsets of X . It will be denoted by $\mathcal{B}(X)$ and its elements called the *Borel* subsets of X .

A measure for which all Borel sets are measurable is a *Borel measure*. It is *Borel regular* if for every $A \subset X$ there exists a Borel $B \supset A$ with $\mu(B) = \mu(A)$.

Theorem 2 (Carathéodory criterion)

Let (X, d) be a metric space and μ a measure on X which is additive on separated sets. That is, whenever $A, B \subset X$ with

$$\inf \{ d(x, y) : x \in A, y \in B \} > 0,$$

we have

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Then μ is a Borel measure.

Proof. Let $C \subset X$ be closed and $E \subset X$ with $\mu(E) < \infty$. We need to show

$$\mu(E) \geq \mu(E \cap C) + \mu(E \setminus C).$$

For each $j \in \mathbb{N}$ let

$$E_j = \{ x \in E : \frac{1}{j+1} < \text{dist}(x, C) \leq \frac{1}{j} \}$$

and

$$E_0 = \{ x \in E : \text{dist}(x, C) > 1 \}.$$

Since C is closed,

$$E \setminus C = E_0 \cup \bigcup_{j \in \mathbb{N}} E_j.$$

Moreover, the E_j with j odd are pairwise separated so

$$\mu(E) \geq \mu\left(\bigcup_{j \text{ odd}} E_j\right) = \sum_{j \text{ odd}} \mu(E_j)$$

and so the sum is convergent. Similarly the sum over even indices is convergent and so

$$\sum_{j \geq n} \mu(E_j) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore

$$\begin{aligned} \mu(E) &\geq \mu\left(E \cap C \cup \bigcup_{j=0}^n E_j\right) \\ &= \mu(E \cap C) + \mu\left(\bigcup_{j=0}^n E_j\right) \\ &\geq \mu(E \cap C) + \mu(E \setminus C) - \sum_{j > n} \mu(E_j) \\ &\rightarrow \mu(E \cap C) + \mu(E \setminus C), \end{aligned}$$

using the additivity on separated sets for the equality and countable sub-additivity for the second inequality.

Definition 5 (Carathéodory construction)

Let (X, d) be a metric space, F a set of subsets of X and $\zeta : F \rightarrow [0, \infty]$. For each $\delta > 0$ and $A \subset X$ define

$$\psi_\delta(A) = \inf \sum_{S \in G} \zeta(S),$$

where the infimum is taken over all countable

$$G \subset \{S \in F : \text{diam}(S)\} < \delta$$

such that

$$A \subset \bigcup_{S \in G} S.$$

Finally, define $\psi(A) = \sup_{\delta > 0} \psi_\delta(A)$.

For any $\delta > 0$, ψ_δ is a measure, as is ψ . *Theorem 8* shows that ψ is a Borel measure on X . Indeed, if $\text{dist}(A, B) > \delta$ then

$$\psi_\delta(A \cup B) \geq \psi_\delta(A) + \psi_\delta(B).$$

If F consists only of Borel sets then ψ is Borel regular.

Remark 2

The fact that $\psi_{\delta'} \leq \psi_\delta$ whenever $\delta' \geq \delta$ implies that

$$\psi(A) = \lim_{\delta \rightarrow 0} \psi_\delta(A).$$

Definition 6

We define some properties of a measure μ on a topological space X .

1. μ is locally finite if every point in X has a neighbourhood of finite measure.
 2. μ is σ -finite if there exist measurable $X_i \subset X$ with $\mu(X_i) < \infty$ and $X = \bigcup_{i \in \mathbb{N}} X_i$.
 3. μ is finite if $\mu(X) < \infty$.
 4. A Borel regular measure μ is a Radon measure if
 1. $\mu(K) < \infty$ for all compact $K \subset X$,
 2. $\mu(A) = \sup\{\mu(K) : K \subset A \text{ compact}\}$ for all Borel $A \subset X$.
 3. $\mu(A) = \inf\{\mu(U) : U \supset A \text{ open}\}$ for all Borel $A \subset X$.
-

Definition 7 (Almost everywhere)

Let μ be a measure on a set X . A property of points in X holds μ almost everywhere (or μ -a.e.) if the set of points for which the property doesn't hold has μ measure zero.

Definition 8 (Push forward)

Let X, Y be sets, μ be a measure on X and let $f: X \rightarrow Y$. The *push forward* of μ under f , written $f_{\#}\mu$ is defined by

$$f_{\#}\mu(S) = \mu(f^{-1}(S)).$$

Definition 9 (Lebesgue measure)

The *Lebesgue measure* on \mathbb{R}^n , denoted \mathcal{L}^n , is defined using the Carathéodory construction with F the set of cubes and $\zeta(Q) = \text{vol}(Q)$. Its measurable sets are called the *Lebesgue measurable* subsets of \mathbb{R}^n .

The following lemma is very useful.

Lemma 2

Let μ be a finite measure on a set X and let \mathcal{S} be a set of μ -measurable subsets of X . There exists disjoint $S_i \in \mathcal{S}$ such that any $S \in \mathcal{S}$ with

$$S \subset X \setminus \bigcup_{i \in \mathbb{N}} S_i$$

satisfies $\mu(S) = 0$.

In particular, if each μ -measurable subset of X of positive measure contains an element of \mathcal{S} of positive measure then we can decompose almost all of X into countably many disjoint elements of \mathcal{S} .

Proof. We find the S_i by induction. First let $\mathcal{S}^1 \subset \mathcal{S}$ be countable and disjoint such that

$$\mu(\cup \mathcal{S}^1) \geq \sup\{\mu(\cup \mathcal{S}') : \mathcal{S}' \subset \mathcal{S} \text{ countable and disjoint}\} - 1/1.$$

Now let \mathcal{M}_2 be the set of all $\mathcal{S}' \subset \mathcal{S}$ that are countable, disjoint and disjoint from $\cup \mathcal{S}^1$. Let $\mathcal{S}^2 \in \mathcal{M}_2$ be such that

$$\mu(\cup \mathcal{S}^2) \geq \sup\{\mu(\cup \mathcal{S}') : \mathcal{S}' \in \mathcal{M}_2\} - 1/2.$$

Inductively, given countable, disjoint $\mathcal{S}^1, \dots, \mathcal{S}^{i-1}$ such that each \mathcal{S}^j and \mathcal{S}^k are disjoint for $k < j$, let \mathcal{M}_i be the set of all $\mathcal{S}' \subset \mathcal{S}$ that are countable, disjoint and disjoint from $\mathcal{S}^1 \cup \dots \cup \mathcal{S}^{i-1}$. Let $\mathcal{S}^i \in \mathcal{M}_i$ be such that

$$\mu(\cup \mathcal{S}^i) \geq \sup\{\mu(\cup \mathcal{S}') : \mathcal{S}' \in \mathcal{M}_i\} - 1/i.$$

We claim that any $S \in \mathcal{S}$ with

$$S \subset X \setminus \bigcup_{i \in \mathbb{N}} \mathcal{S}^i$$

satisfies $\mu(S) = 0$. If not, let $i \in \mathbb{N}$ be such that $1/i < \mu(S)$. Then $\mathcal{T} := \mathcal{S}^i \cup \{S\} \in \mathcal{M}_i$ and

$$\mu(\cup \mathcal{T}) > \sup\{\mu(\cup \mathcal{S}') : \mathcal{S}' \in \mathcal{M}_i\} - 1/i + 1/i,$$

a contradiction.

1.1 Exercises

Example 1

Usually in measure theory, a measure is defined as a countably additive function defined on a σ -algebra. However, using our definition is simply a convenience rather than a restriction.

Indeed, suppose μ is a countably additive function defined on a σ -algebra Σ of X with $\mu(\emptyset) = 0$. Show that it can be extended to the power set of X by

$$\bar{\mu}(A) = \inf\{\mu(B) : A \subset B \in \Sigma\}$$

and that any $B \in \Sigma$ is μ -measurable. What about

$$\underline{\mu}(A) = \sup\{\mu(B) : A \supset B \in \Sigma\}?$$

Conversely, any measure is countably additive when restricted to any σ -algebra of measurable sets.

Example 2

Let Ω be a set of subsets of a set X . Show that $\Sigma(\Omega)$ is a σ -algebra. Note that it is the smallest σ -algebra of X containing Ω .

Example 3

Show that the following sets are Borel subsets of \mathbb{R} : \mathbb{Q} , $[0, 1)$, the set of points in $[0, 1]$ whose first decimal is even.

Let $f : [0, 1] \rightarrow [0, 1]$. Show that the set of points where f is continuous is a Borel set. What about the set of points where f is differentiable?

Example 4

Let X be a set and $x \in X$. The Dirac measure at x is defined as $\delta_x(A) = 1$ if $x \in A$, $\delta_x(A) = 0$ otherwise. Show that δ_x is a measure on X . What are its measurable sets?

Define the *counting measure* on X to be the cardinality (finite or ∞) of any subset of X . Show that this is a measure. What are its measurable sets?

Example 5

For (X, d) a metric space and $s \geq 0$ the s -dimensional Hausdorff measure on X , denoted \mathcal{H}^s , is defined using the Carathéodory construction with \mathcal{F} the set of all sets and $\zeta(S) = \text{diam}(S)^s$.

1. Show that \mathcal{L}^n and \mathcal{H}^n are non-zero, translation invariant and n -homogenous measures. That is, for any $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $t > 0$, $\mathcal{L}^n(A + x) = \mathcal{L}^n(A)$ and $\mathcal{L}^n(tA) = t^n \mathcal{L}^n(A)$ (and similarly for \mathcal{H}^n).
2. On \mathbb{R}^n show that there exists a $C > 0$ such that $\mathcal{H}^n/C \leq \mathcal{L}^n \leq C\mathcal{H}^n$.
3. Let $f: X \rightarrow Y$ be an L -Lipschitz function between two metric spaces. Show that for any $s \geq 0$ and $A \subset X$,

$$\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^s(A).$$

4. For any metric space X , show that \mathcal{H}^0 is the counting measure on X .
 5. For $0 \leq s < t < \infty$, suppose that $\mathcal{H}^s(A) < \infty$. Show that $\mathcal{H}^t(A) = 0$. Hence there exists a single $0 \leq s \leq \infty$ for which $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = \infty$ for all $t < s$. This t is called the *Hausdorff dimension* of A , denoted $\dim_{\mathbb{H}} A$.
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Example 6 (Cantor set)

The *Cantor set* $K \subset [0, 1]$ is defined as follows. Let $K_0 = [0, 1]$ and for each $i \in \mathbb{N}$ let K_i be obtained from deleting the “middle third” open interval from each of the intervals in K_{i-1} . That is, $K_1 = [0, 1/3] \cup [2/3, 1]$,

$$K_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1],$$

etc. Define $K = \bigcap_{i \in \mathbb{N}} K_i$. Note that K is compact and hence Borel.

1. Show that K is uncountable.
2. Let $s = \log 2 / \log 3$. Show that $0 < \mathcal{H}^s(K) < \infty$.

In particular, K is an uncountable subset of \mathbb{R} with $\mathcal{L}^1(K) = 0$.

Example 7 (Restriction measure)

Give an examples of $S \subset \mathbb{R}^2$ with $\dim_{\mathbb{H}} S = 1$ for which $\mathcal{H}^1|_S$ is:

1. finite,
 2. σ -finite but not finite,
 3. not σ -finite.
-

Example 8 (Vitali set)

The fundamental properties of measures are those given in *Theorem 6*, in particular countable additivity. It is necessary for us to only require this to be true for measurable sets, as can be seen from the existence of non-measurable sets.

Define a *Vitali set* as follows. Consider the equivalence relation \sim on \mathbb{R} defined by $x \sim y$ iff $x - y \in \mathbb{Q}$. By the density of \mathbb{Q} in \mathbb{R} , each equivalence class V_x intersects $[0, 1]$. Therefore, by the axiom of choice(!), we may construct a set $\mathcal{V} \subset [0, 1]$ consisting of exactly one member of each equivalence class.

Show:

1. If $p \neq q$ are rational then $p + \mathcal{V}$ and $q + \mathcal{V}$ are disjoint.
2. $[0, 1] \subset \bigcup \{q + \mathcal{V} : q \in \mathbb{Q} \cap [-1, 1]\} \subset [-1, 2]$.
3. Show that $\mathcal{L}^1(\mathcal{V}) \neq 0$.
4. Deduce that \mathcal{V} is not Lebesgue measurable.

Example 9

Show that the two extensions given in *Example 1* may not agree. For example, after extending Lebesgue measure (restricted to the Borel sets), what are the values of a Vitali set?

Example 10

Let μ be a finite Borel measure on a metric space X . Prove that for every Borel $B \subset X$,

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ closed}\} \tag{1.2}$$

and

$$\mu(B) = \inf\{\mu(U) : U \supset B \text{ open}\}. \tag{1.3}$$

Property (1.2) is called *inner regularity by closed sets* and (1.3) is called *outer regularity by open sets*.

Hint: observe that it suffices to show that all Borel sets satisfy (1.2). Show that the set

$$\{B \subset X : B \text{ and } X \setminus B \text{ are inner regular by closed sets}\}$$

is a σ -algebra that contains all closed subsets of X .

Show that a σ -finite μ is inner regular by closed sets. Show that a σ -finite μ is outer regular by open sets if there exist open sets $U_i \subset X$ with $\mu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $X = \bigcup_{i \in \mathbb{N}} U_i$. Give an example of a σ -finite μ that is *not* outer regular by open sets.

Example 11

Let X be a complete and separable metric space. Show that any finite Borel measure on X is a Radon measure.

Hint: a metric space is compact if and only if it is complete and totally bounded.

Definition 10 (Simple function)

Let μ be a measure on a set X . A *simple function* is any function of the form

$$\sum_{i=1}^m a_i \chi_{A_i},$$

where each $a_i \in \mathbb{R}$ and the $A_i \subset X$ are disjoint μ -measurable sets. We treat $0 \cdot \infty = 0$.

Definition 11 (Integral)

Let μ be a measure on a set X and let $f: X \rightarrow \mathbb{R}^+$. The (lower) integral of f with respect to μ is

$$\int f \, d\mu := \sup \left\{ \sum_{i=1}^m a_i \mu(A_i) : s = \sum_{i=1}^m a_i \chi_{A_i} \leq f, s \text{ simple} \right\}.$$

Definition 12 (Measurable function)

Let μ be a measure on a set X . A function $f: X \rightarrow \mathbb{R}$ is μ -measurable if $f^{-1}((a, \infty))$ is μ -measurable for every $a \in \mathbb{R}$.

For $f: X \rightarrow \mathbb{R}$ measurable, let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ (both μ -measurable), so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. If one of $\int_X f^+ \, d\mu$ and $\int_X f^- \, d\mu$ are finite, we say that f is μ -integrable and we define the integral of f with respect to μ to be

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

If only $\int_X f^- \, d\mu < \infty$ (respectively $\int_X f^+ \, d\mu < \infty$) we write $\int_X f \, d\mu = \infty$ (respectively $\int_X f \, d\mu = -\infty$).

Let X be a topological space. A function $f: X \rightarrow \mathbb{R}$ is a Borel function if $f^{-1}((a, \infty))$ is a Borel set for every $a \in \mathbb{R}$.

There are some simple properties of the integral to check, such as linearity and monotonicity. See [Example 14](#).

Linear combinations of measurable functions are measurable, as are limits of measurable functions. See [Example 13](#).

Theorem 3 (Fatou's lemma)

Let μ be a measure on a set X and $f_k: X \rightarrow [0, \infty]$ μ -measurable. Then

$$\int_X \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k \, d\mu.$$

Proof. Let

$$s = \sum_{i=1}^m a_i \chi_{A_i}$$

be a simple function with

$$s \leq \liminf f_k.$$

for each $x \in A_i$ and each $1 \leq i \leq m$ and let $0 < t < 1$. For each $1 \leq i \leq m$, the sets

$$G_{k,i} := \{x \in A_i : f_k(x) \geq ta_i \text{ for all } j \geq k\}$$

monotonically increase to A_i as k increases. Therefore

$$\int f_k \, d\mu \geq \sum_{i=1}^m ta_i \mu(G_{k,i}) \rightarrow \sum_{i=1}^m ta_i \mu(A_i).$$

and hence

$$\sum_{i=1}^m ta_i \mu(A_i) \leq \liminf_{k \rightarrow \infty} \int f_k \, d\mu.$$

Since $0 < t < 1$ is arbitrary, the conclusion follows.

Remark 3 (Reverse Fatou)

Suppose that there exists $g \geq 0$ with $\int g \, d\mu < \infty$ and $f_k \leq g$ for all k . Then

$$\limsup_{k \rightarrow \infty} \int f_k \, d\mu \geq \limsup_{k \rightarrow \infty} \int_X f_k \, d\mu.$$

Indeed, this follows by applying Fatou's lemma to $g - f_k$.

Theorem 4 (Monotone convergence theorem)

Let μ be a measure on a set X and $f_k: X \rightarrow [0, \infty]$ μ -measurable. Suppose that for every $x \in X$ and all $k \in \mathbb{N}$, $f_{k+1}(x) \geq f_k(x)$. Then

$$\lim_{k \rightarrow \infty} \int f_k \, d\mu = \int \lim_{k \rightarrow \infty} f_k \, d\mu.$$

Proof. The monotonicity of the integral gives \leq whilst Fatou's lemma gives \geq .

Theorem 5 (Dominated convergence theorem)

Let μ be a measure on X and $f_n: X \rightarrow \mathbb{R}$ μ -measurable such that $f_n \rightarrow f$ pointwise. Suppose that there exists measurable $g: X \rightarrow [0, \infty]$ with $\int g \, d\mu < \infty$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Then

$$\int f_n \, d\mu \rightarrow \int f \, d\mu.$$

Proof. Observe that for all $n \in \mathbb{N}$, $|f - f_n| \leq 2g$ and that $\limsup |f - f_n| = 0$. Then by the reverse Fatou lemma,

$$\left| \int f \, d\mu - \int f_n \, d\mu \right| \leq \int |f - f_n| \, d\mu \rightarrow 0.$$

2.1 Exercises

Example 12

For μ a measure on a set X , let $f: X \rightarrow \mathbb{R}$ be measurable, respectively Borel. Show that the pre-image of any Borel $B \subset \mathbb{R}$ is μ -measurable, respectively Borel. Compare this to the definition of a continuous function.

Example 13

Let μ be a measure on X and for each $i \in \mathbb{N}$ let $f_i: X \rightarrow \mathbb{R}$ be μ -measurable. Show that the functions

$$\limsup_{i \rightarrow \infty} f_i \quad \text{and} \quad \liminf_{i \rightarrow \infty} f_i$$

are μ -measurable.

Show that a linear combination of μ -measurable functions is μ -measurable. Show that a countable (pointwise) sum of μ -measurable functions is μ -measurable.

Example 14

There are some simple properties of the integral to check:

1. If $f \leq g$ μ -a.e. then

$$\int f \, d\mu \leq \int g \, d\mu;$$

2. The integral with respect to μ is a linear operator;
3. If $S \subset X$ is μ -measurable then

$$\int_X f \, d\mu = \int_S f \, d\mu + \int_{X \setminus S} f \, d\mu;$$

4. $|\int f \, d\mu| \leq \int |f| \, d\mu;$

5. etc...

Example 15

Show that $f: X \rightarrow \mathbb{R}$ is μ -measurable if and only if

$$\mu(E) \geq \mu(E \cap f^{-1}((-\infty, a))) + \mu(E \cap f^{-1}((b, \infty)))$$

for every $E \subset X$ and $a < b \in \mathbb{Q}$.

Example 16

State and prove a reverse monotone convergence theorem for monotonically decreasing sequences of functions.

Example 17

Show that the Fatou lemma is false if the functions are not uniformly bounded below.

Show that the reverse Fatou lemma is false if the sequence is not bounded above by an integrable g .

Show that the monotone convergence theorem is false if the sequence does not monotonically increase.

Example 18

Let X, Y be sets, μ a measure on X and $f: X \rightarrow Y$. Show that $f_{\#}\mu$ is a measure on Y . If X, Y are topological spaces and μ, f are Borel, show that $f_{\#}\mu$ is a Borel measure on Y .

THE DANIELL INTEGRAL

Let μ be a measure on a set X and let L be a set of real valued μ -measurable functions on X . The formula

$$T(f) := \int_X f \, d\mu$$

defines an operator on L . Moreover, it has the following two properties:

- T is *monotonic*: for all $f, g \in L$ with $g \leq f$, $T(g) \leq T(f)$;
- T is *continuous with respect to monotone convergence*: if $f_i \in C(X)$ monotonically increase to f then $T(f_i) \rightarrow T(f)$.

In the next theorem we see that these two properties completely characterise the Lebesgue integral. That is, we could equivalently develop a theory of integration (and hence measure) by beginning with operators on sets of functions.

Definition 13 (Lattice of functions)

Let X be a set. A lattice of functions on X is a non-empty set L of functions $X \rightarrow \mathbb{R}$ which satisfies the following conditions: for any $c \in \mathbb{R}^+$ and any $f, g \in L$, $f + g$, cf , $\inf\{f, g\}$ and $\inf\{f, c\}$ all belong to L and if $g \leq f$ then $f - g \in L$ too. Note that any vector space of functions closed under \inf is a lattice.

If L is a lattice we let

$$L^+ = \{f \in L : f \geq 0\}.$$

We say a $T: L \rightarrow \mathbb{R}$ is

1. *Linear* if, for all $f, g \in L$ and $a, b \in \mathbb{R}^+$,

$$T(af + bg) = aT(f) + bT(g);$$

2. *Monotonic* if, for all $f, g \in L$ with $g \leq f$,

$$T(g) \leq T(f);$$

3. *Continuous with respect to monotone convergence* if, for all $f_i \in L$ that monotonically increase to f ,

$$T(f_i) \rightarrow T(f).$$

4. *Bounded* if for every $f \in L$,

$$\sup\{T(g) : 0 \leq g \leq f\} < \infty.$$

A T satisfying the first three items is called a monotone Daniell integral (it necessarily satisfies the fourth item). A T satisfying items 1, 3 and 4 is called a Daniell integral.

Theorem 6

Let L be a lattice on X and let $T: L \rightarrow \mathbb{R}$ be a monotone Daniell integral. Then there exists a measure μ on X for which each $f \in L^+$ is μ -measurable such that

$$T(f) = \int f \, d\mu$$

for all $f \in L$.

Proof. First note that, for any $f \in L^+$,

$$T(f) \geq T(0 \cdot f) = 0.$$

For $A \subset X$ we say that a sequence $f_i \in L^+$ suits A if the f_i monotonically increase and

$$\lim_{i \rightarrow \infty} f_i(x) \geq 1 \quad \forall x \in A.$$

Define

$$\mu(A) = \inf \left\{ \lim_{i \rightarrow \infty} T(f_i) : f_i \text{ suits } A \right\}.$$

Then μ is a measure on X . Indeed, $\mu(\emptyset) = 0$ and μ is monotonic. If $A_j \subset X$ for each $j \in \mathbb{N}$ and $f_i^j \in L^+$ suit A_j then

$$g_i := \sum_{j=1}^i f_i^j$$

suits $\cup_j A_j$ and

$$T(g_i) = \sum_{j=1}^i T(f_i^j) \leq \sum_{j \in \mathbb{N}} \lim_{i \rightarrow \infty} T(f_i^j) = \sum_{j \in \mathbb{N}} \mu(A_j).$$

Next we show that each $f \in L^+$ is μ -measurable. By *Exercise 15*, it suffices to show, for every $E \subset X$ and $0 \leq a < b \in \mathbb{R}$ that, for

$$A := f^{-1}((-\infty, a)), \quad B := f^{-1}((b, \infty)),$$

we have

$$\mu(E) \geq \mu(E \cap A) + \mu(E \cap B).$$

Suppose g_i suit E and let

$$h = \frac{\inf\{f, b\} - \inf\{f, a\}}{b - a}, \quad k_i = \inf\{g_i, h\}.$$

Then $0 \leq k_{i+1} - k_i \leq g_{i+1} - g_i$ for all i and

$$h(x) = 1 \text{ whenever } f(x) \geq b, \quad h(x) = 0 \text{ whenever } f(x) \leq a.$$

Then k_i suit B and $g_i - k_i$ suit A . Therefore

$$\lim_{i \rightarrow \infty} T(g_i) = \lim_{i \rightarrow \infty} T(k_i) + T(g_i - k_i) \geq \mu(B) + \mu(A).$$

Finally we show that

$$T(f) = \int f \, d\mu \quad f \in L^+. \quad (3.1)$$

First suppose that $A \subset X$, f_i suit A and $g \in L^+$ satisfies $g \leq \chi_A$. Then $h_i = \inf\{f_i, g\}$ monotonically increase to g and so

$$T(g) = \lim_{i \rightarrow \infty} T(h_i) \leq \lim_{i \rightarrow \infty} T(f_i).$$

Consequently,

$$T(g) \leq \mu(A). \quad (3.2)$$

Now fix $f \in L^+$ and for each $t \in \mathbb{R}^+$ let $f_t = \inf\{f, t\}$. For a moment fix $\epsilon > 0$. Then, for each $k \in \mathbb{N}$,

$$0 \leq f_{k\epsilon}(x) - f_{(k-1)\epsilon}(x) \leq \epsilon \quad \forall x \in X, \quad (3.3)$$

$$f_{k\epsilon}(x) - f_{(k-1)\epsilon}(x) = \epsilon \quad \text{whenever } f(x) \geq k\epsilon \quad (3.4)$$

and

$$f_{k\epsilon}(x) - f_{(k-1)\epsilon}(x) = 0 \quad \text{whenever } f(x) \leq (k-1)\epsilon. \quad (3.5)$$

Note that, for any $k \in \mathbb{N}$, $(f_{k\epsilon} - f_{(k-1)\epsilon})/\epsilon$ suits

$$\{x : f(x) \geq k\epsilon\}$$

and so

$$T(f_{k\epsilon} - f_{(k-1)\epsilon}) \geq \epsilon \mu(\{x : f(x) \geq k\epsilon\}). \quad (3.6)$$

By (3.3) and (3.4) respectively,

$$\begin{aligned} \epsilon \mu(\{x : f(x) \geq k\epsilon\}) &\geq \int f_{(k+1)\epsilon} - f_{k\epsilon} \, d\mu \\ &\geq \epsilon \mu(\{x : f(x) \geq (k+1)\epsilon\}). \end{aligned} \quad (3.7)$$

Finally by (3.2),

$$\epsilon \mu(\{x : f(x) \geq (k+1)\epsilon\}) \geq T(f_{(k+2)\epsilon} - f_{(k+1)\epsilon}) \quad (3.8)$$

Combining (3.6), (3.8) and (3.7) gives

$$T(f_{k\epsilon} - f_{(k-1)\epsilon}) \geq \int f_{(k+1)\epsilon} - f_{k\epsilon} \, d\mu \geq T(f_{(k+2)\epsilon} - f_{(k+1)\epsilon}).$$

Summing from 1 to i gives

$$T(f_{i\epsilon}) \geq \int f_{(i+1)\epsilon} - f_\epsilon \, d\mu \geq T(f_{(i+2)\epsilon} - f_{2\epsilon}).$$

Since $f_{i\epsilon}$ monotonically increases to f ,

$$T(f) \geq \int f - f_\epsilon \, d\mu \geq T(f - f_\epsilon).$$

Therefore, since f_ϵ monotonically decreases to 0, $f - f_\epsilon$ increases to f and so this gives (3.1).

Note that, if $f \in L$, then $f^+, f^- \in L^+$ and so

$$T(f) = T(f^+) - T(f^-) = \int f^+ d\mu - \int f^- d\mu = \int f d\mu.$$

Observation 1

For any measure μ satisfying the conclusion of [Theorem 6](#), the value of $\mu(\{f > t\})$, for $f \in L^+$ and $t > 0$, is uniquely determined by the values of T on $L^+.$ *

Proof. For any $t > 0$ and $0 < h < t$, observe that the functions

$$g_h := \frac{\inf\{f, t+h\} - \inf\{f, t\}}{h}$$

converge pointwise to the characteristic function of $f^{-1}((t, \infty))$ and are bounded above by $2f$. Therefore, by [\[mu_rep_T\]](#) and the dominated convergence theorem,

$$\mu(\{x : f(x) > t\}) = \lim_{h \rightarrow 0} h^{-1}T(\inf\{f, t+h\} - \inf\{f, t\}).$$

The Banach–Alaoglu theorem (see [Example 26](#)) motivates us to consider representations of Borel measures by elements of the dual of a Banach space, namely of $C(X)$. To use [Theorem 6](#), we must upgrade the pointwise convergence in the hypotheses to uniform convergence in $C(X)$. Recall that $C_c(X)$ is the set of all continuous functions on X with compact support, and that pointwise monotonic convergence in $C_c(X)$ implies uniform convergence (see [Example 22](#)). Therefore, for any monotonic $T \in C_c(X)'$, [Theorem 6](#) produces a measure μ that represents T . If X is a metric space, then all compact sets are measurable with respect to μ . We next show, on locally compact spaces, how to obtain a Borel measure.

Lemma 3

Let X be a locally compact metric space and suppose that τ is a finite measure on X . There exists a unique Radon measure ν on X that agrees with τ on $\mathcal{K}(X)$.

Proof. Let \mathcal{U} be the set of open subsets of X . For each $U \in \mathcal{U}$ define

$$\nu(U) = \sup\{\tau(K) : K \subset U \text{ compact}\}.$$

Since τ is monotone, for any $K \in \mathcal{K}(X)$, $\nu(K^\circ) \leq \tau(K)$, for K° the interior of K .

Since X is locally compact, for any $U \in \mathcal{U}$ and $K \in \mathcal{K}(X)$ with $K \subset U$, there exists $V \in \mathcal{U}$ with $\bar{V} \in \mathcal{K}(X)$ and

$$K \subset V \subset \bar{V} \subset U.$$

Consequently,

$$\nu(U) = \sup\{\nu(V) : V \in \mathcal{U}, V \subset U, \bar{V} \in \mathcal{K}(X)\}. \tag{3.9}$$

Also, since τ is a measure and X is a metric space,

$$\tau(K) = \inf\{\tau(S) : K \subset S^\circ, S \in \mathcal{K}(X)\}. \tag{3.10}$$

For each $A \subset X$ define

$$\mu(A) = \inf\{\nu(U) : U \supset A \text{ open}\}.$$

Note that both ν and μ are monotone and give value 0 to the empty set. To see that μ is a measure, let $A_i \subset X$ and let $U_i \supset A_i$ be open. We must show that

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \leq \sum_{i \in \mathbb{N}} \mu(A_i).$$

It suffices to show that

$$\nu\left(\bigcup_{i \in \mathbb{N}} U_i\right) \leq \sum_{i \in \mathbb{N}} \nu(U_i).$$

By (3.9), for any $\epsilon > 0$, there exists $V \subset \bigcup_i U_i$ with compact closure such that

$$\nu(V) \geq \nu\left(\bigcup_{i \in \mathbb{N}} U_i\right) - \epsilon$$

Since V has compact closure, it is contained in the union of finitely many U_i . Therefore, it suffices to show that ν is finitely sub-additive.

Let $U, V \in \mathcal{U}$ and $K \subset U \cup V$ be compact. Let $W = U \cap V$ and

$$K_U := \{x \in K : d(x, U \setminus W) \leq d(x, V \setminus W)\}$$

and

$$K_V := \{x \in K : d(x, V \setminus W) \leq d(x, U \setminus W)\}.$$

Then K_U, K_V are closed subsets of K with $K = K_U \cup K_V$ and $K_U \subset U$ and $K_V \subset V$. Since τ is finitely sub-additive,

$$\tau(K) \leq \tau(K_U) + \tau(K_V)$$

and so

$$\nu(U \cup V) \leq \nu(U) + \nu(V).$$

Therefore, ν is finitely sub-additive by induction. As shown above, this implies that μ is a measure.

To see that μ is a Borel measure, if $A, B \subset X$ are separated, then there exist separated open sets $U \supset A, V \supset B$. Since τ is finitely additive, $\nu(U \cap V) = \nu(U) + \nu(V)$ and so μ is additive on separated sets. By construction, μ is Borel regular. Also, (3.9) shows that open sets are inner regular by compact sets. Combining this with outer regularity by open sets shows that μ is a Radon measure.

To see that μ agrees with τ on $\mathcal{K}(X)$, note that for any $K \in \mathcal{K}(X)$ and $U \in \mathcal{U}$ with $K \subset U$,

$$\tau(K) \leq \nu(U) = \mu(U)$$

and so $\tau(K) \leq \mu(K)$. For the other inequality,

$$\begin{aligned} \mu(K) &= \inf\{\nu(U) : U \supset K \text{ open}\} \\ &\leq \inf\{\nu(S^\circ) : S \in \mathcal{K}(X), K \subset S^\circ\} \\ &\leq \inf\{\tau(S) : S \in \mathcal{K}(X), K \subset S^\circ\} = \tau(K) \end{aligned}$$

by (3.10).

Finally, if μ_1, μ_2 are two Radon measures that agree with τ on $\mathcal{K}(X)$, then for any open $U \subset X$,

$$\begin{aligned}\mu_1(U) &= \sup\{\mu_1(K) : K \subset U, K \in \mathcal{K}(X)\} \\ &= \sup\{\mu_2(K) : K \subset U, K \in \mathcal{K}(X)\} = \mu_2(U).\end{aligned}$$

Since μ_1, μ_2 are both Borel regular, they must agree.

Theorem 7 (Riesz representation theorem)

Let X be a locally compact metric space and let $T \in C_c(X)'$ be monotone. Then there exists a unique Radon measure μ such that

$$T(f) = \int f \, d\mu \quad \forall f \in C_c(X). \tag{3.11}$$

Proof. Observe that T is a monotone Daniell integral on $L = C_c(X)$. By *Theorem 6*, there exists a measure τ on X for which (3.11) holds with μ replaced by τ . By *Lemma 3*, there exists a Radon measure μ that agrees with τ on $\mathcal{K}(X)$. For any $f \in C_c(X)$, the value of

$$\int f \, d\tau$$

is determined by the value of τ on compact sets, and so

$$\int f \, d\tau = \int f \, d\mu,$$

so that (3.11) holds. *Observation 1* implies that, on a locally compact space, the measure of any compact set is uniquely determined by (3.11). Thus any Radon measure satisfying (3.11) is uniquely determined.

The Riesz representation theorem allows us to identify the set of finite Radon measures on a locally compact space X with the set of monotonic elements of $C_0(X)'$. For B a Banach space, a sequence $T_n \in B'$ weak* converges to $T \in B'$ if $T_n(x) \rightarrow T(x)$ for every $x \in B$. In $C_0(X)'$, this translates to the following.

Definition 14 (Weak* convergence)

Let X be a locally compact metric space. A sequence μ_n of finite Radon measures on X weak* converges to a finite Radon measure μ if, for every $f \in C_c(X)$,

$$\int_X f \, d\mu_n \rightarrow \int_X f \, d\mu.$$

By the Banach-Alaoglu theorem (see *Example 26*), the unit ball of B' is weak* compact. Since the weak* limit of a sequence of monotonic operators on $C_0(X)$ is monotonic, we have the following.

Theorem 8

Let X be a locally compact metric space and μ_n a sequence of Radon measures with uniformly bounded total measures. There exists a finite Radon measure μ on X and subsequence μ_{n_k} that weak* converges to μ .

3.1 Exercises

Example 19

Let L be a lattice of functions on X . For any $f \in L$ show that $f^+, f^- \in L^+$

Example 20

Let L be a lattice on X and T a monotone Daniell integral on L . Give an example to show that there may be more than one measure satisfying (3.11) (recall *Example 9*). Compare to *Observation 1*.

Example 21

Let L be a lattice on X and T a Daniell integral on L . Define T^+, T^- on L^+ by

$$T^+(f) = \sup\{T(g) : g \in L^+, g \leq f\}$$

and

$$T^-(f) = -\inf\{T(g) : g \in L^+, g \leq f\}.$$

1. There show that T^+, T^- are monotone Daniell integrals on L .
2. If $f, g \in L^+$ with $g \leq f$ then $f \geq f - g \in L^+$ and so

$$T(g) - T^-(f) \leq T(g) + T(f - g) \leq T(g) + T^+(f).$$

3. Deduce that $T = T^+ - T^-$.

Example 22

Let K be a compact metric space and suppose that $f_i \in C(K)$ monotonically increase to $f \in C(K)$. Show that $f_i \rightarrow f$ uniformly.

Example 23

Let c be the set of all $f: \mathbb{N} \rightarrow \mathbb{R}$ such that $\lim_j f(x_j)$ exists and define $T: c \rightarrow \mathbb{R}$ by $T(f) = \lim_j f(x_j)$. Let \mathcal{C} be the set of all bounded $f: \mathbb{N} \rightarrow \mathbb{R}$. Equip c and \mathcal{C} with the supremum norm.

1. Observe that T is linear and continuous on c and hence can be extended by the Hahn-Banach theorem to a linear and continuous element of \mathcal{C} . (Such an extension is called a *Banach limit*.)
2. Show that any finite Borel measure on \mathbb{N} is a convergent sum of Dirac masses.
3. Hence show that there is no Borel measure μ on \mathbb{R} such that $T_\mu = T$.

Example 24

Adapt the previous exercise to show that the Riesz representation theorem is false in non locally compact metric spaces.

Example 25

Let X be a metric space. Show that any $T \in C_c(X)'$ is a Daniell integral.

1. Let $T^+, T^- \in C_c(X)'^+$ be obtained from *Example 21*.
 2. Show that $\|T\| = \|T^+\| + \|T^-\|$.
 3. By *Theorem 7*, any $T \in C_0(X)$ can be identified with two measures μ^+ and μ^- and hence with a signed measure (recall *Example 60*). Show that this identification is an isometric isomorphism.
-

Example 26 (Banach–Alaoglu theorem)

Let B be a separable Banach space and \mathcal{D} a countable dense subset of B . Suppose that $T_n \in B'$ satisfy $\|T_n\| \leq M$ for some $M > 0$.

1. Show that there exists a subsequence T_{n_j} and a $T \in B'$ such that $T_{n_j}(d) \rightarrow T(d)$ for each $d \in \mathcal{D}$.
2. Deduce that, for any $x \in B$, $T_{n_j}(x) \rightarrow T(x)$.

That is, closed and bounded subsets of B' are weak* compact.

COVERING THEOREMS

We will use $B(x, r)$ to denote the closed ball in a metric space X centred at $x \in X$ with radius $r \geq 0$. Since the centre and radius of a ball are not uniquely defined by its elements, formally by a “ball” we mean a pair $(x, r) \in X \times (0, \infty)$, but in practice we mean the set of its elements.

Lemma 4 ($5r$ covering lemma)

Let X be a metric space and \mathcal{B} an arbitrary collection of closed balls of uniformly bounded radii. There exists a disjoint sub-collection $\mathcal{B}' \subset \mathcal{B}$ such that any $B \in \mathcal{B}$ intersects a ball $B' \in \mathcal{B}'$ with

$$\text{rad } B' \geq \text{rad } B/2.$$

In particular,

$$\bigcup_{B \in \mathcal{B}'} 5B \supset \bigcup_{B \in \mathcal{B}} B.$$

Here, $5B$ denotes the ball with the same centre as B and 5 times the radius.*

Proof. For each $n \in \mathbb{Z}$ let

$$\mathcal{B}_n = \{B \in \mathcal{B} : 2^n \leq \text{rad } B < 2^{n+1}\}.$$

Since the balls in \mathcal{B} have uniformly bounded radii, the \mathcal{B}_n are empty for all $n > N$, for some $N \in \mathbb{N}$. Let \mathcal{B}'_N be a maximal disjoint sub-collection of \mathcal{B}_N . That is, the elements of \mathcal{B}'_N are disjoint elements of \mathcal{B}_N and if $B \in \mathcal{B}_N$, there exists a $B' \in \mathcal{B}'_N$ with $B \cap B' \neq \emptyset$. (In general such a maximal collection exists by Zorn’s lemma. See also *Exercise 42*.) Let \mathcal{B}'_{N-1} be a maximal collection such that $\mathcal{B}'_N \cup \mathcal{B}'_{N-1}$ is a disjoint collection. Repeat this for each $i \in \mathbb{N}$, obtaining a maximal collection \mathcal{B}'_{N-i} such that $\mathcal{B}'_N \cup \dots \cup \mathcal{B}'_{N-i}$ is a disjoint collection, and set $\mathcal{B}' = \bigcup_{n \leq N} \mathcal{B}'_n$.

Now suppose that $B \in \mathcal{B}$, say $B \in \mathcal{B}_n$. Then by construction there exists $B' \in \mathcal{B}'_m$ for some $m \geq n$ with $B \cap B' \neq \emptyset$. In particular, $\text{rad } B' \geq \text{rad } B/2$.

The final statement of the lemma follows from the triangle inequality.

Definition 15 (Vitali cover)

Let X be a metric space and $S \subset X$. A *Vitali cover* of S is a collection \mathcal{B} of closed balls such that, for each $x \in S$ and each $\epsilon > 0$, there exists a ball $B \in \mathcal{B}$ with $\text{rad } B < \epsilon$ and $x \in B$.

Proposition 1

Let X be a metric space, $S \subset X$ and suppose that \mathcal{B} is a Vitali cover of S . Then there exists a disjoint $\mathcal{B}' \subset \mathcal{B}$ such that, for every finite $I \subset \mathcal{B}'$,

$$S \setminus \bigcup_{B \in I} B \subset \bigcup_{B \in \mathcal{B}' \setminus I} 5B.$$

In particular, if $\mathcal{B}' = \{B_1, B_2, \dots\}$ is countable (for example, if X is separable), then

$$S \setminus \bigcup_{i=1}^n B_i \subset \bigcup_{i>n} 5B_i$$

for each $n \in \mathbb{N}$.

Proof. Note that we may suppose \mathcal{B} consists of balls with uniformly bounded radii. Let \mathcal{B}' be a disjoint sub-collection of \mathcal{B} obtained from *Lemma 4*. If $I \subset \mathcal{B}'$ is finite then

$$C := \bigcup_{B \in I} B$$

is closed. Therefore, if $x \in S \setminus C$, since \mathcal{B} is a Vitali cover of S , there exists $B \in \mathcal{B}$ with $x \in B$ such that $B \cap C = \emptyset$. However, B must intersect some $B' \in \mathcal{B}'$ with $\text{rad } B' \geq \text{rad } B/2$, and so $x \in 5B'$. That is, x belongs to

$$\bigcup_{B \in \mathcal{B}' \setminus I} 5B,$$

as required.

Definition 16 (Doubling measure)

A Borel measure μ on a metric space X is a *doubling measure* if there exists a $C_\mu \geq 1$ such that

$$0 < \mu(2B) \leq C_\mu \mu(B) < \infty$$

for all balls $B \subset X$.

Remark 4

Note that, for any $m \geq 2$,

$$\mu(mB) \leq C_\mu^{\log_2 m} \mu(B).$$

Lebesgue measure is a doubling measure.

Theorem 9 (Vitali covering theorem)

Let μ be a doubling measure on a metric space X and let \mathcal{B} be a Vitali cover of a set $S \subset X$. There exists a countable disjoint $\mathcal{B}' \subset \mathcal{B}$ such that

$$\mu\left(S \setminus \bigcup_{B \in \mathcal{B}'} B\right) = 0.$$

Proof. First note that it suffices to prove the result for S bounded, say S is contained in some ball \tilde{B} . We may also suppose that each $B \in \mathcal{B}$ is a subset of $2\tilde{B}$.

Let \mathcal{B}' be a disjoint sub-collection of \mathcal{B} obtained from [Proposition 1](#). Note that \mathcal{B}' is countable. Indeed, for each $m \in \mathbb{N}$, at most $m\mu(2\tilde{B})$ balls $B \in \mathcal{B}'$ can satisfy $\mu(B) > 1/m$.

Enumerate $\mathcal{B}' = \{B_1, B_2, \dots\}$. Since the B_i are disjoint subsets of $2\tilde{B}$,

$$\sum_{i>n} \mu(B_i) \rightarrow 0.$$

By the conclusion of [Proposition 1](#),

$$S \setminus \bigcup_{i=1}^n B_i \subset \bigcup_{i>n} 5B_i$$

for each $n \in \mathbb{N}$. Since μ is doubling, $\mu(5B_i) \leq C\mu(B_i)$ for each $i \in \mathbb{N}$ and so

$$\mu\left(S \setminus \bigcup_{i=1}^n B_i\right) \leq C \sum_{i>n} \mu(B_i) \rightarrow 0,$$

as required.

Definition 17 (Hardy–Littlewood maximal function)

Let μ be a Borel measure on a metric space X and $f: X \rightarrow \mathbb{R}$ μ -measurable. Suppose that $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and all $r > 0$.

Define the *Hardy–Littlewood maximal function* of f by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f| \, d\mu.$$

By [Example 29](#), the maximal function is a Borel function.

Theorem 10 (Hardy–Littlewood maximal inequality)

Let μ be a doubling measure on a metric space X . There exists a $C > 0$ such that, for any $f: X \rightarrow \mathbb{R}$ and $\lambda > 0$,

$$\mu(\{x : Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \int |f| \, d\mu. \tag{4.1}$$

Proof. For $\lambda > 0$, let

$$S = \{x \in X : Mf(x) > \lambda\}$$

and, for each $R > 0$, let S_R be those $x \in X$ for which there exists $0 < r < R$ such that

$$\int_{B(x, r)} |f| \, d\mu > \lambda\mu(B(x, r)). \tag{4.2}$$

Similarly to [Example 29](#), each S_R is a Borel set. Moreover, the S_R monotonically increase to S . For a moment fix $R > 0$. Let \mathcal{B} be the collection of balls $B(x, r)$ with $x \in S_R$ and $0 < r < R$ that satisfy (4.2) and let \mathcal{B}' satisfy the conclusion of [Lemma 4](#). Then

$$\begin{aligned} \int_X |f| \, d\mu &\geq \sum_{B \in \mathcal{B}'} \int_B |f| \, d\mu \\ &> \sum_{B \in \mathcal{B}'} \lambda \mu(B) \\ &\geq \frac{1}{C_\mu^{\log_2 5}} \sum_{B \in \mathcal{B}'} \lambda \mu(5B) \\ &\geq \frac{\lambda}{C_\mu^{\log_2 5}} \mu(S_R). \end{aligned}$$

In particular, (4.1) holds for $C = C_\mu^{\log_2 5}$.

Theorem 11 (Lebesgue differentiation theorem)

Let μ be a doubling measure on a metric space X and $f: X \rightarrow \mathbb{R}$ with $\int f \, d\mu < \infty$. For μ -a.e. $x \in X$,

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f - f(x)| \, d\mu \rightarrow 0$$

as $r \rightarrow 0$. Such an x is called a Lebesgue point of f .*

Proof. First note that the theorem is true if f is continuous.

Fix $\epsilon > 0$ and let $g: X \rightarrow \mathbb{R}$ be continuous with

$$\int |f - g| \, d\mu < \epsilon$$

(such a g exists by [Example 55](#)). Let

$$B = \{x \in X : |f(x) - g(x)| \geq \sqrt{\epsilon}\},$$

so that $\mu(B) < \sqrt{\epsilon}$.

If

$$S = \{x : M(f - g) > \sqrt{\epsilon}\},$$

then by [Theorem 10](#),

$$\mu(S) \leq \frac{C}{\sqrt{\epsilon}} \|f - g\|_1 < C\sqrt{\epsilon}.$$

Moreover, if $x \notin S$,

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f - g| \, d\mu \leq \sqrt{\epsilon}$$

for all $r > 0$. In particular, since g is continuous at x ,

$$\limsup_{r \rightarrow 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f - g(x)| \, d\mu \leq \sqrt{\epsilon}.$$

Therefore, if $x \notin B$,

$$\limsup_{r \rightarrow 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f - f(x)| \, d\mu \leq 2\sqrt{\epsilon}.$$

We are now done; repeat the above for a countable collection of $\epsilon \rightarrow 0$. The corresponding $B \cup S$ monotonically decrease to a set of measure zero. The set of $x \in X$ that does not belong to infinitely many of the $B \cup S$ has full measure, and for such an x ,

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f - f(x)| \, d\mu = 0.$$

Corollary 1 (Lebesgue density theorem)

Let μ be a doubling measure on a metric space X and let $S \subset X$ be μ -measurable with $\mu(S) < \infty$. Then

$$\lim_{r \rightarrow 0} \frac{\mu(S \cap B(x, r))}{\mu(B(x, r))}$$

equals 1 for μ -a.e. $x \in S$ and 0 for μ -a.e. $x \notin S$. Such an x for which the limit equals 1 is called a density point of S .

4.1 Exercises

Example 27

Let X be a separable metric space. Show that for any collection of balls, there exists a maximal disjoint sub-collection.

Example 28

Show that the $5r$ covering Lemma may not be true if the radii are not uniformly bounded.

Example 29

Let μ be a finite Borel measure on a metric space (X, d) and let $x_n \rightarrow x \in X$ such that $d(x, x_n)$ is a decreasing sequence. Let $U(y, r)$ denote the open ball centred on y with radius r .

1. Show that, for any $r > 0$,

$$U(x, r) \setminus U(x_n, r)$$

decreases to the empty set.

2. Deduce that $y \mapsto \mu(U(y, r))$ is lower semi-continuous.
3. Give an example to show that $y \mapsto \mu(U(y, r))$ may not be continuous.
4. Show that, for any $y \in X$,

$$\mu(B(y, r)) = \lim_{\mathbb{Q} \ni q \downarrow r} \mu(U(y, q)).$$

5. Deduce that, for any $r > 0$, $y \mapsto \mu(B(y, r))$ is a Borel function.

6. Show that the Hardy–Littlewood maximal function is equivalently defined by taking the supremum over all rational $r > 0$.
 7. Deduce that the Hardy–Littlewood maximal function is a Borel function.
-

Example 30

Let μ be a doubling measure on a metric space X and let $S \subset X$ with $\mu(S) < \infty$. Suppose that there exists a μ -measurable $S' \supset S$ with $\mu(S') = \mu(S)$. Show that

$$\lim_{r \rightarrow 0} \frac{\mu(S \cap B(x, r))}{\mu(B(x, r))} = 1$$

for μ -a.e. $x \in S$.

Example 31

Prove that on \mathbb{R}^n , $\mathcal{L}^n = c\mathcal{H}^n$ for some $c > 0$. There are two ways to prove this (recall [Example 5](#)).

Let μ, ν be two finite Borel measures on a set X with $\mu \ll \nu$ and suppose that ν is doubling. Show that the Radon–Nikodym derivative of μ with respect to ν is given by

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

for ν -a.e. $x \in X$.

Example 32

For $f = \chi_{[0,1]}$, show that Mf does not have finite integral.

DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

The regularity of a Lipschitz function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is very interesting. Of course, Lipschitz functions are continuous, but they may not be differentiable everywhere. However, it is quite easy to convince yourself that they cannot be non-differentiable on quite a large set. The question to quantify how large the non-differentiability set of a Lipschitz function can be was one of the motivating questions of Lebesgue's development of measure theory.

Definition 18 (Total variation of a measure)

Let $f: [a, b] \rightarrow \mathbb{R}$. The *total variation* of f , $Vf: [a, b] \rightarrow [0, \infty]$ is defined by

$$Vf(x) = \sup \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where the supremum ranges over all $a = t_0 < t_1 < \dots < t_n = b$.

If $Vf(b) < \infty$, f is said to have *bounded variation (BV)*.

Definition 19 (Absolutely continuous function)

A function $f: [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous (AC)* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that, for any intervals $(a_1, b_1), (a_2, b_2), \dots \subset [a, b]$ with $\sum_i |b_i - a_i| < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

Note that AC functions are BV, and Lipschitz functions are AC (see [Example 33](#)). Also, if f is BV then Vf and $Vf - f$ are non-decreasing. If f is AC then so are Vf and $Vf - f$, see [Example 34](#).

Theorem 12 (Lebesgue)

Let $f: [a, b] \rightarrow \mathbb{R}$ be absolutely continuous. Then f is differentiable \mathcal{L}^1 almost everywhere. Moreover, for any $x > y \in [a, b]$,

$$f(x) - f(y) = \int_y^x f' dx.$$

Proof. By [Example 34](#) it suffices to assume that f is non-decreasing. In this case define a measure μ on $[a, b]$ using the Carathéodory construction with F the set of compact intervals and $\zeta([c, d]) = f(d) - f(c)$. This defines a finite Borel measure such that $\mu([c, d]) = f(d) - f(c)$ for all intervals $[c, d] \subset [a, b]$. Indeed, for any $\delta > 0$, we may cover $[c, d]$ by finitely many intervals $[c, c_1], [c_1, c_2], \dots, [c_k, d]$ of width $\delta' \leq \delta$, showing

$$\mu([c, d]) \leq \sum f(c_{i+1}) - f(c_i) = f(d) - f(c).$$

The reverse inequality holds because f is non-decreasing.

Note that $\mu \ll \mathcal{L}^1$. Indeed, given $\epsilon > 0$, let $\delta > 0$ be given by the definition of f being absolutely continuous. If $\mathcal{L}^1(N) = 0$, we may cover N by countably many closed intervals I_i such that $\sum_i \mathcal{L}^1(I_i) < \delta$. In particular $\sum_i f(I_i) < \epsilon$ and hence $\mu(N) < \epsilon$. Therefore,

$$\mu = \int \frac{d\mu}{d\mathcal{L}^1} d\mathcal{L}^1,$$

with $d\mu/d\mathcal{L}^1 \in L^1(\mathcal{L}^1)$.

By the Lebesgue differentiation theorem, for any Lebesgue point x of $d\mu/d\mathcal{L}^1$,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f(x+t) - f(x)}{t} &= \lim_{t \rightarrow 0} \frac{\mu([x+t, x])}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_x^{x+t} \frac{d\mu}{d\mathcal{L}^1} d\mathcal{L}^1 \\ &= \frac{d\mu}{d\mathcal{L}^1}(x). \end{aligned}$$

Theorem 13 (Rademacher)

Any Lipschitz $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable \mathcal{L}^n almost everywhere.

Proof. For notational simplicity, we prove the case $n = 2$.

For each $y \in \mathbb{R}$, $x \mapsto f(x, y)$ is a Lipschitz function $\mathbb{R} \rightarrow \mathbb{R}$ and so is differentiable \mathcal{L}^1 -a.e. That is, for every y , $\partial_1 f(x, y)$ exists for \mathcal{L}^1 -a.e. x . By Fubini's theorem, $\partial_1 f$ exists \mathcal{L}^2 -a.e. Similarly, $\partial_2 f$ exists almost everywhere too.

Fix $\epsilon > 0$. For $D \in \mathbb{Q}^2$ and $j \in \mathbb{N}$ let

$$X_{D,j} = \{x : |f(x + he_i) - f(x) - D_i h| < \epsilon|h|, \forall 0 < |h| < 1/j, i = 1, 2\}.$$

These are Borel sets. Further, for $D \in \mathbb{Q}^2$, if

$$|\partial_1 f(x) - D_1| < \epsilon/2 \quad \text{and} \quad |\partial_2 f(x) - D_2| < \epsilon/2,$$

then $x \in X_{D,j}$ for sufficiently large j . That is,

$$X^\epsilon = \bigcup_{D \in \mathbb{Q}^2} \bigcup_{j \in \mathbb{N}} X_{D,j}$$

is a set of full measure.

Fix $D \in \mathbb{Q}^2$ and $j \in \mathbb{N}$. Let x be a density point of $X_{D,j}$. Let $R > 0$ such that

$$\mathcal{L}^n(B(x, r) \cap X_{D,j}) \geq (1 - \epsilon^n) \mathcal{L}^n(B(x, r))$$

for all $0 < r < R$. In particular, for every $y \in B(x, r)$ there exists $y' \in X_{D,j}$ with

$$\|y - y'\| < \epsilon \|y - x\|.$$

Now let $r < \min\{R, 1/j\}$ and $\|x - y\| < r$. Set $h = y - x$, $\tilde{y} = x + \pi_1 y$ and $\tilde{y} \in X_{D,j}$ with

$$\|\tilde{y} - y'\| < \epsilon \|x - \tilde{y}\| \leq \epsilon \|x - y\|.$$

Also let y', y'' lie on the same vertical line as \tilde{y} such that y', \tilde{y} have the same vertical component as do y'', \tilde{y} . Then, since $x \in X_{D,j}$,

$$|f(\tilde{y}) - f(x) - D_1 h_1| < \epsilon |h_1| = \epsilon \|x - \tilde{y}\| \leq \epsilon \|x - y\|; \quad (5.1)$$

Since f is Lipschitz,

$$|f(\tilde{y}) - f(y')| \leq L \|\tilde{y} - y'\| \leq L \epsilon \|x - y\|; \quad (5.2)$$

Since $\tilde{y} \in X_{D,j}$,

$$|f(y'') - f(y') - D_2 h_2| \leq \epsilon \|y' - y''\| \leq \epsilon \|x - y\|; \quad (5.3)$$

Since f is Lipschitz,

$$|f(y'') - f(y)| \leq L \|y'' - y\| = L \|y' - \tilde{y}\| \leq L \|\tilde{y} - \tilde{y}\| \leq \epsilon L \|x - y\|. \quad (5.4)$$

By combining (5.1), (5.2) (5.3) and (5.4),

$$|f(y) - f(x) - D \cdot h| \leq 2(1 + L)\epsilon \|x - y\|.$$

This is true for all y with $\|x - y\| < r$ and for any density point x of the full measure set X^ϵ . That is, for \mathcal{L}^n -a.e x . Taking a countable intersection over $\epsilon \rightarrow 0$ concludes the proof.

5.1 Exercises

Example 33

Exercise 48. Prove that Lipschitz functions are AC and that AC functions are BV.

Example 34

Exercise 49. Let $f: [a, b] \rightarrow \mathbb{R}$ be BV. Show that Vf and $Vf - f$ are non-decreasing. If f is AC then show that Vf and $Vf - f$ are AC.

Example 35

Exercise 50. Show that any monotonic $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous except at countably many points.

Example 36

Exercise 51. In this exercise we will show that monotonic functions are differentiable almost everywhere.

Let $f: [a, b] \rightarrow \mathbb{R}$ be non-decreasing. For each $x \in (a, b)$ let

$$\underline{D}f(x) = \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \overline{D}f(x) = \limsup_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Observe that the set of $x \in (a, b)$ where f is not differentiable at x is the countable union, over $p < q \in \mathbb{Q}$, of the sets

$$B_{p,q} := \{x \in (a, b) : \underline{D}f(x) < p < q < \overline{D}f(x)\}.$$

We now fix $p < q \in \mathbb{Q}$.

1. Let

$$\mathcal{B} = \{[x, x+h] : f(x+h) - f(x) < ph\}.$$

Note that \mathcal{B} satisfies the hypotheses of the Vitali covering theorem (recall *Theorem 43*). Let \mathcal{B}' be a disjoint sub-cover obtained from the Vitali covering theorem with respect to Lebesgue measure and let $S = \cup \mathcal{B}'$. Prove that

$$\mathcal{L}^1(f(B_{p,q} \cap S)) \leq p\mathcal{L}^1(B_{p,q} \cap S).$$

Note: this is the step where we require f to be monotonic.

2. Similarly, prove that $\mathcal{L}^1(f(B_{p,q} \cap S)) \geq q\mathcal{L}^1(B_{p,q} \cap S)$.
 3. Deduce that f is differentiable almost everywhere.
 4. Deduce that a BV function is differentiable almost everywhere.
-

However, BV functions do not satisfy the fundamental theorem of calculus:

Example 37

Exercise 52. Recall the definition of the Cantor set from *Exercise 6*. Define the *Cantor function* $f: [0, 1] \rightarrow [0, 1]$ as follows. For each $n \in \mathbb{N}$, define $f_n: [0, 1] \rightarrow [0, 1]$ by

$$f(x) = \left(\frac{3}{2}\right)^n \mathcal{L}^1([0, x] \cap C_n).$$

Show that the f_n converge uniformly on $[0, 1]$ to a monotonic, continuous function f . For each $x \in [0, 1] \setminus C$, show that $f'(x) = 0$.

Thus, f is monotonic and hence BV, has derivative 0 almost everywhere, but does not satisfy the fundamental theorem of calculus.

Example 38

Exercise 53. In lectures we proved that the derivative of any AC function is an absolutely continuous measure. Prove the converse: for any finite, absolutely continuous measure μ on $[0, \infty)$, show that

$$f(x) := \int_0^x \frac{d\mu}{d\mathcal{L}^1} d\mathcal{L}^1 = \mu([0, x])$$

defines an absolutely continuous function.

Up to now, we have considered points where functions *are* differentiable. We now consider points of non-differentiability (which are much more interesting).

Example 39

Show that the Cantor function is not differentiable at any point of the Cantor set.

Example 40

Let $N \subset [0, 1]$ satisfy $\mathcal{L}^1(N) = 0$.

1. For each $n \in \mathbb{N}$, iteratively construct a countable collection of open intervals \mathcal{O}_n such that, for each $n \in \mathbb{N}$,
- N is contained in the union of \mathcal{O}_n ;
 - for every $I \in \mathcal{O}_n$ there exists $J \in \mathcal{O}_{n-1}$ with $I \subset J$;
 - for each $I \in \mathcal{O}_{n-1}$,

$$\mathcal{L}^1(I \cap \cup\{J : J \in \mathcal{O}_n\}) < 2^{-n}|I|.$$

2. Let

$$S = \bigcap_{n \in \mathbb{N}} \bigcup_{m > n} \cup\{J : J \in \mathcal{O}_m\},$$

the “limsup” of the \mathcal{O}_n (S is the set of points that are contained in infinitely many intervals from the \mathcal{O}_n). In particular, $S \supset N$.

For each $x \in [0, 1] \setminus S$, let $N(x)$ be the largest n for which there exists $I \in \mathcal{O}_n$ with $x \in I$. Define $P(x) = 1$ if $N(x)$ is even, $P(x) = 0$ otherwise. Finally, for each $x \in [0, 1]$ define

$$f(x) = \mathcal{L}^1(\{t \in [0, x] : P(t) = 1\}).$$

Show that f is Lipschitz, monotonic, and not differentiable at any point of N . Hint: show that $\underline{D}f(x) = 0$ and $\overline{D}f(x) = 1$ for each $x \in N$.

HAUSDORFF MEASURE AND DENSITIES

Recall the definition of Hausdorff measure from *Example 5*:

Definition 20 (Hausdorff measure)

For X a metric space, $s \geq 0$ and $A \subset X$, let

$$\mathcal{H}_\delta^s(A) = \inf \sum_{S \in G} \text{diam}(S),$$

where the infimum is taken over all countable

$$G \subset \{S \in F : \text{diam}(S)\} < \delta$$

such that

$$A \subset \bigcup_{S \in G} S.$$

Finally, define $\mathcal{H}^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A)$.

We are interested in the measure $\mathcal{H}^s|_S$, for $S \subset X$ some \mathcal{H}^s -measurable set with $\mathcal{H}^s(S) < \infty$. In particular, we require some counterpart to the Lebesgue density theorem, but, of course, $\mathcal{H}^s|_X$ may not be locally finite.

Definition 21 (Hausdorff density)

Let X be a metric space, $A \subset X$ and $s \geq 0$. The upper and lower *Hausdorff densities* of A are

$$\Theta^{*,s}(A, x) = \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(A \cap B(x, r))}{(2r)^s}$$

and

$$\Theta_*^s(A, x) = \liminf_{r \rightarrow 0} \frac{\mathcal{H}^s(A \cap B(x, r))}{(2r)^s}.$$

Lemma 5

Let X be a metric space, $s \geq 0$ and $A \subset X$ with $\mathcal{H}^s(A) < \infty$. Then

$$2^{-s} \leq \Theta^{*,s}(A, x) \leq 1$$

for \mathcal{H}^s -a.e. $x \in A$.

Proof. The set of $x \in A$ with $\Theta^{*,s}(A, x) < 2^{-s}$ is a countable union of the sets

$$S_\delta := \{x \in A : \mathcal{H}^s(A \cap B(x, r)) < (1 - \delta)r^s \forall 0 < r < \delta\}.$$

Thus, for the first inequality, it suffices to show that $\mathcal{H}^s(S_\delta) = 0$ for all $\delta > 0$.

Fix $\delta, \epsilon > 0$. We may cover S_δ by sets E_1, E_2, \dots such that, for each $i \in \mathbb{N}$, $\text{diam } E_i < \epsilon$, $S_\delta \cap E_i \neq \emptyset$ and

$$\sum_{i \in \mathbb{N}} \text{diam } E_i^s \leq \mathcal{H}^s(S_\delta) + \epsilon.$$

For each $i \in \mathbb{N}$ let $x_i \in S_\delta \cap E_i$ and set $r_i = \text{diam } E_i$. Then

$$\begin{aligned} \mathcal{H}^s(S_\delta) &\leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(S_\delta \cap E_i) \leq \sum_{i \in \mathbb{N}} \mathcal{H}^s(A \cap B(x_i, r_i)) \\ &\leq (1 - \delta) \sum_{i \in \mathbb{N}} \text{diam } E_i^s \leq (1 - \delta)(\mathcal{H}^s(S_\delta) + \epsilon). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary and $\delta > 0$, this implies $\mathcal{H}^s(S_\delta) = 0$, as required.

For the second inequality, since \mathcal{H}^s is Borel regular (see [Example 42](#)), it suffices to assume that A is Borel. As before, given $\delta > 0$, it suffices to prove that

$$S := \{x \in A : \Theta^{*,s}(A, x) > 1 + \delta\}$$

satisfies $\mathcal{H}^s(S) = 0$. Fix $\epsilon > 0$ and let $U \supset S$ be open with

$$\mathcal{H}^s(A \cap U) \leq \mathcal{H}^s(S) + \epsilon$$

(which exists by the outer regularity of the measure $\mathcal{H}^s|_A$). Let \mathcal{B}_ϵ be the collection of balls B centred at a point of S with $\text{rad } B < \epsilon$ such that $B \subset U$ and

$$\mathcal{H}^s(A \cap B) > (1 + \delta)(2 \text{rad } B)^s. \tag{6.1}$$

This is a Vitali cover of S . Let \mathcal{B}'_ϵ be obtained from [Proposition 1](#).

Since $\mathcal{H}^s(S) < \infty$, S is separable (see [Example 44](#)) and so $\mathcal{B}'_\epsilon = \{B_1, B_2, \dots\}$ is countable and the conclusion of [Proposition 1](#) states that

$$S \setminus \bigcup_{i \in \mathbb{N}} B_i \subset \bigcup_{i > n} 5B_i$$

for each $n \in \mathbb{N}$. Since $\text{diam } B_i < \epsilon$ for each $i \in \mathbb{N}$, the B_i and $5B_i$ may be used to estimate $\mathcal{H}_{10\epsilon}^s(S)$. For each $n \in \mathbb{N}$ we obtain

$$\begin{aligned} \mathcal{H}_{10\epsilon}^s(S) &\leq \sum_{i \in \mathbb{N}} (2 \text{rad } B_i)^s + \sum_{i > n} (10 \text{rad } B_i)^s \\ &\leq \sum_{i \in \mathbb{N}} \frac{\mathcal{H}^s(A \cap B_i)}{1 + \delta} + 5^s \sum_{i > n} \frac{\mathcal{H}^s(A \cap B_i)}{1 + \delta} \end{aligned}$$

where the second inequality follows by (6.1). Since the B_i are disjoint and $\mathcal{H}^s(A) < \infty$, the second term converges to 0 as $n \rightarrow \infty$. Since the B_i are subsets of U we obtain

$$\mathcal{H}_{10\epsilon}^s(S) \leq \frac{\mathcal{H}^s(A \cap U)}{1 + \delta} \leq \frac{\mathcal{H}^s(S) + \epsilon}{1 + \delta}.$$

Since $\epsilon > 0$ is arbitrary, this implies $\mathcal{H}^s(S) \leq \mathcal{H}^s(S)/(1 + \delta)$ and hence $\mathcal{H}^s(S) = 0$, as required.

Lemma 6

Let X be a metric space, $s \geq 0$ and let $A \subset X$ be \mathcal{H}^s -measurable with $\mathcal{H}^s(A) < \infty$. Then

$$\Theta^{*,s}(A, x) = 0$$

for \mathcal{H}^s -a.e. $x \notin A$.

Proof. It suffices to show that, for $t > 0$, the set

$$S = \{x \in X \setminus A : \Theta^{*,s}(A, x) > t\}$$

satisfies $\mathcal{H}^s(S) = 0$. Fix $\epsilon > 0$. Since A is \mathcal{H}^s -measurable, $\mathcal{H}^s|_A$ is Borel regular. Therefore, since $\mathcal{H}^s|_A(S) = 0$, there exists an open $U \supset S$ with

$$\mathcal{H}^s(A \cap U) = \mathcal{H}^s|_A(U) < \epsilon.$$

For each $x \in S$ and $\delta > 0$ there exists a ball B centred on x with $\text{rad } B < \delta$ such that

$$\frac{\mathcal{H}^s(A \cap B)}{(2 \text{rad } B)^s} > t.$$

By *Lemma 4* there exists a disjoint collection \mathcal{B} of such balls such that

$$S \subset \bigcup_{B \in \mathcal{B}} 5B.$$

Since $\mathcal{H}^s(A) < \infty$, A is separable and each of these balls contains a point of A , \mathcal{B} is countable. Therefore

$$t\mathcal{H}_{5\delta}^s(S) \leq t \sum_{B \in \mathcal{B}} (2 \text{rad } 5B)^s < 5^s \sum_{B \in \mathcal{B}} \mathcal{H}^s(A \cap B) \leq 5^s \mathcal{H}^s(A \cap U) < 5^s \epsilon.$$

Since $\delta, \epsilon > 0$ are arbitrary, this completes the proof.

6.1 Exercises

Example 41

Let $\mathcal{V} \subset [0, 1]$ be a Vitali set as constructed in *Exercise 8*.

1. Show that, for any Borel $B \subset \mathcal{V}$, $\mathcal{L}^1(B) = 0$.
2. Deduce that $\mathcal{L}^1([0, 1] \setminus \mathcal{V}) = 1$ and hence, if C is a Borel set with

$$[0, 1] \setminus \mathcal{V} \subset C \subset [0, 1],$$

then $\mathcal{L}^1(C) = 1$.

3. Hence show that $\mathcal{L}^1(\mathcal{V} \cap C) = \mathcal{L}^1(\mathcal{V}) > 0$.

Note however that we cannot deduce the value of $\mathcal{L}^1(\mathcal{V})$ from our construction in [Example 8](#). Indeed, for any $\epsilon > 0$, that construction may produce a $\mathcal{V} \subset [0, \epsilon]$.

Example 42

Let X be a metric space and $s \geq 0$.

1. Show that \mathcal{H}^s is Borel regular. Hint: first show that in the definition of \mathcal{H}^s , we may take F to be the collection of *closed* sets.
 2. We are usually interested in $\mathcal{H}^s|_A$ for some $A \subset X$. Show that for *any* $A \subset X$, $\mathcal{H}^s|_A$ is a Borel measure.
 3. Now assume that $A \subset X$ is \mathcal{H}^s -measurable with $\mathcal{H}^s(A) < \infty$. Show that $\mathcal{H}^s|_A$ is Borel regular. Hint: show that there exist Borel sets $B \supset A \supset B'$ with $\mathcal{H}^s(B \setminus B') = 0$.
 4. Show that $\mathcal{H}^s|_A$ may not be Borel regular if A is not \mathcal{H}^s measurable. Hint: consider [Example 41](#).
-

Example 43

In this exercise we construct the four corner Cantor set. Let $K_0 = [0, 1]^2$. Let K_1 be the “four corners” of K_0 of side length $1/4$. That is

$$K_1 = [0, 1/4]^2 \cup [3/4, 1]^2 \cup [0, 1/4] \times [3/4, 1] \cup [3/4, 1] \times [0, 1/4].$$

Inductively, K_n is constructed by taking the four corners of side length $1/4^n$ of all the squares of K_{n-1} . Finally let $K = \bigcap_{n \in \mathbb{N}} K_n$, a compact set. Show that $0 < \mathcal{H}^1(K) < \infty$.

Example 44

For $s \geq 0$ let X be a metric space with $\mathcal{H}^s(X) < \infty$. Show that X is separable.

Example 45

Show that [Lemma 5](#) and [Lemma 6](#) may be false if A has only σ -finite \mathcal{H}^s measure.

RECTIFIABLE SETS AND APPROXIMATE TANGENT PLANES

Rectifiable sets are the measure theoretic counterpart to manifolds.

Definition 22 (Rectifiable set)

A \mathcal{H}^n -measurable set $E \subset \mathbb{R}^m$ is n -rectifiable if there exist Lipschitz $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\mathcal{H}^n \left(E \setminus \bigcup_{i \in \mathbb{N}} f_i(\mathbb{R}^n) \right) = 0.$$

We will show that n -rectifiable sets possess a unique approximate n -dimensional tangent plane at almost every point.

Given $V \in G(m, n)$, $a \in \mathbb{R}^m$ and $0 < s < 1$ define the cone around V centred at a with aperture s as

$$C(a, V, s) = \{x \in \mathbb{R}^m : \text{dist}(x - a, V) < s\|x - a\|\}.$$

Definition 23 (Approximate tangent plane)

Let $A \subset \mathbb{R}^m$ and $a \in A$. A $V \in G(m, n)$ is an approximate tangent plane to A at a if

$$\limsup_{r \rightarrow 0} \frac{\mathcal{H}^n(A \cap B(a, r))}{r^n} > 0 \tag{7.1}$$

and, for every $0 < s < 1$,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A \cap B(a, r) \setminus C(a, V, s))}{r^n} = 0. \tag{7.2}$$

Rademacher's theorem gives a candidate for the approximate tangent plane to a rectifiable set. There are three steps required to prove that the derivative is indeed an approximate tangent plane: show that the derivative has full rank at almost every point; prove the density condition (7.1); and show that the sets from other parametrisations of the rectifiable set do not destroy the approximation by a tangent plane at almost every point.

The second and third steps follow from the results of the previous section. For the first step we use the following.

Lemma 7 (Easy Sard's theorem)

If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz then

$$\mathcal{H}^n(\{f(x) : \text{rank } Df(x) < n\}) = 0.$$

Proof. Let $L = \text{Lip } f$. Fix $0 < R < \infty$, $\delta, \epsilon > 0$ and let

$$A = \{x \in B(x, R) : \text{rank } Df(x) < n\}.$$

For $x \in A$ let

$$W_x = f(x) + Df(x)(\mathbb{R}^n).$$

Then for sufficiently small $0 < r_x < \delta$,

$$f(B(x, r_x)) \subset B(f(x), Lr_x) \cap \{y : \text{dist}(y, W_x) < \epsilon r_x\}.$$

Since $\text{rank } Df(x) < n$, the set on the right hand side can be covered by $(L/\epsilon)^{n-1}$ cubes of side length ϵr_x .

Since A is covered by balls of the form $B(x, r_x/5)$, there exists a disjoint collection of balls $B(x_i, r_i/5)$ such that A is covered by the union of the $B(x_i, r_i)$. Then

$$f(A) \subset f\left(\bigcup_{i \in \mathbb{N}} B(x_i, r_i)\right) \subset \bigcup_{i \in \mathbb{N}} f(B(x_i, r_i)).$$

By the previous argument, each factor of the right hand side is covered by $(L/\epsilon)^{n-1}$ cubes of side length ϵr_i . Thus

$$\mathcal{H}_{2\delta}^n(f(A)) \leq \sum_{i \in \mathbb{N}} \left(\frac{L}{\epsilon}\right)^{n-1} (\epsilon r_i)^{n-1} = L^{n-1} \epsilon \sum_{i \in \mathbb{N}} r_i^n.$$

However, the $B(x_i, r_i/5)$ are disjoint subsets of $B(0, R + \delta) \subset \mathbb{R}^n$ and so

$$\sum_{i \in \mathbb{N}} \left(\frac{r_i}{5}\right)^n \leq (R + \delta)^n.$$

Since $\epsilon > 0$ is arbitrary, this implies that $\mathcal{H}_{2\delta}^n(f(A)) = 0$ and hence $\mathcal{H}^n(f(A)) = 0$. Taking a countable union over $R \rightarrow \infty$ completes the proof.

Lemma 8

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz and $S \subset \mathbb{R}^n$. Suppose that there exists a $\delta > 0$ such that, for each $x, y \in S$,

$$\|f(x) - f(y)\| \geq \delta \|x - y\|.$$

Then $f(S)$ has a unique approximate tangent plane at \mathcal{H}^n almost every point.*

Proof. By *Lemma 7*, we may suppose $\text{rank } Df(x) = n$ for every $x \in S$. By *Lemma 5*, we may suppose $\Theta^{*,n}(E, f(x)) > 0$ for every $x \in S$. Fix $x \in S$ and $0 < s < 1$. There exists $\epsilon > 0$ such that

$$\|f(y) - f(x) - Df(x)(y - x)\| < s \|y - x\|$$

for all $y \in B(x, \epsilon) \cap S$. Moreover, if $y \in S \setminus B(x, \epsilon)$ then

$$\|f(y) - f(x)\| \geq \delta \epsilon.$$

That is, if $a = f(x)$ and $b = f(y)$ with $\|a - b\| \leq \delta\epsilon$ and $V = a + Df(x)(\mathbb{R}^n)$,

$$\text{dist}(b - a, V) < s\|y - x\| \leq s\|b - a\|/\delta.$$

Therefore, V is an approximate tangent plane to $f(S)$ at a .

This approximate tangent plane is unique at any density point x of S . Indeed, if $V' \neq V$, let $v \in V \setminus V'$ and let $0 < s < 1$ be such that $C(f(x), \mathbb{R}v, s) \cap C(f(x), V', s) = \{0\}$. Since $\text{rank } Df(x) = n$ and x is a density point of S , for sufficiently small $r > 0$ there exists $y \in S \cap B(x, r)$ such that $B(f(y), sr) \cap C(f(x), V', s) = \emptyset$ and

$$\frac{\mathcal{H}^n(B(f(y), sr) \cap S)}{r^n} \geq \delta s.$$

In particular, V' is not an approximate tangent to $f(S')$ at x .

Theorem 14

Let $E \subset \mathbb{R}^m$ be n -rectifiable with $\mathcal{H}^n(E) < \infty$. Then for \mathcal{H}^n -a.e. $x \in E$, E has a unique approximate tangent plane at x .

Proof. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be one of the Lipschitz functions as in the definition of a rectifiable set and let $S = f^{-1}(E)$. It suffices to prove that E has a unique approximate tangent plane at $f(x)$ for \mathcal{L}^n -a.e. $x \in S$.

By [Lemma 7](#), we may suppose that $\dim Df(\mathbb{R}^n) = n$ for all $x \in S$. Fix such an x and let $0 < \epsilon < \|Df(x)^{-1}\|/2$. There exists $\delta > 0$ such that

$$\|f(y) - f(x) - Df(x)(y - x)\| < \epsilon\|y - x\|$$

for all $y \in B(x, \delta) \cap S$. In particular, by the triangle inequality,

$$\|f(y) - f(x)\| > \epsilon\|y - x\|/2.$$

Therefore, the sets

$$S_\epsilon := \{x \in G : \|f(y) - f(x)\| > \epsilon\|y - x\| \forall y \in B(x, \epsilon)\}$$

are Borel and monotonically increase to S as $\epsilon \rightarrow 0$. Therefore it suffices to prove the result for \mathcal{L}^n -a.e. x in some fixed S_ϵ . Cover S_ϵ by finitely many balls B_1, B_2, \dots, B_N of radius ϵ . It suffices to prove the result for \mathcal{L}^n -a.e. x in some fixed $S' := S_\epsilon \cap B_i$.

However, S' satisfies the hypotheses of [Lemma 8](#) and so $f(S')$ has a unique approximate tangent at \mathcal{H}^n almost every point. To see that this tangent is a unique approximate tangent to E at \mathcal{H}^n almost every point, we simply use [Lemma 6](#): for \mathcal{H}^n -a.e. $x \in f(S')$, $\Theta^{*,n}(E \setminus f(S'), x) = 0$.

In [Theorem 16](#) we will see that the converse to [Theorem 14](#) holds.

For $V \in G(n, m)$, write π_V for the orthogonal projection onto V and equip $G(n, m)$ with the metric $d(V, W) = \|\pi_V - \pi_W\|$. We will consider \mathcal{L}^n on an element of $G(n, m)$.

Lemma 9

Let $f: \mathbb{R}^n \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz and let $S \subset \mathbb{R}^n$ satisfy $\mathcal{L}^n(S) > 0$. For $\epsilon > 0$ suppose that there exists an invertible linear $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that, for all $x, y \in S$,

$$\|f(x) - f(y) - L(x - y)\| < \frac{\epsilon}{\|L^{-1}\|} \|x - y\|.$$

Then for any $V \in G(n, m)$ with $\|(\pi_V|_{L(\mathbb{R}^n)})^{-1}\|^{-1} \geq 2\epsilon$, $\mathcal{L}^n(\pi_V(f(S))) > 0$.*

Proof. For any $V \in G(n, m)$,

$$\|\pi_V(f(x) - f(y)) - \pi_V(L(x - y))\| < \epsilon \|L^{-1}\|^{-1} \|x - y\|$$

and so, if $\|(\pi_V|_{L(\mathbb{R}^n)})^{-1}\|^{-1} \geq 2\epsilon$,

$$\begin{aligned} \|\pi_V(f(x) - f(y))\| &\geq \|\pi_V(L(x - y))\| - \epsilon \|L^{-1}\|^{-1} \|x - y\| \\ &\geq \|(\pi_V|_{L(\mathbb{R}^n)})^{-1}\|^{-1} \|L(x - y)\| - \epsilon \|L(x - y)\| \\ &\geq \epsilon \|L(x - y)\| \\ &\geq \epsilon \|L^{-1}\| \|x - y\|. \end{aligned}$$

Thus $\pi_V \circ f$ has Lipschitz inverse on S and hence $\mathcal{L}^n(\pi_V(f(S))) > 0$ by *Example 5*.

Corollary 2

Let $E \subset \mathbb{R}^m$ be n -rectifiable with $\mathcal{H}^n(E) > 0$. Then there exists $W \in G(m - n, m)$ such that $\pi_V(E) > 0$ for all $V \in G(n, m)$ with $V \cap W = \{0\}$.*

Remark 5

The set of V that satisfy the conclusion of *Corollary 2* is very large; try some examples in reasonable dimensions.

Proof. Since $E \subset \mathbb{R}^m$ is rectifiable with $\mathcal{H}^n(E) > 0$, there exists a Lipschitz $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\mathcal{H}^n(E \cap f(\mathbb{R}^n)) > 0$. In particular, $S := f^{-1}(E)$ satisfies $\mathcal{L}^n(S) > 0$. By *Lemma 7*, for \mathcal{L}^n -a.e. $x \in S$, $Df(x)$ is injective.

Fix $\epsilon > 0$. For $M > 0$, the set of invertible $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\|L^{-1}\| < M$ may be covered by countably many sets of diameter ϵ/M . Varying $M \in \mathbb{N}$, we see that \mathcal{L}^n almost all of S is covered by countably many sets of the form

$$\{x \in S : \|Df(x) - L\| < \epsilon/2 \|L^{-1}\|\}.$$

Moreover, each of these sets may be covered by countably many sets of the form

$$\{x \in S : \|f(x) - f(y) - L(x - y)\| < \epsilon \|x - y\| \|L^{-1}\| \forall y \in B(x, \epsilon)\}.$$

Finally, these sets may be covered by countably many sets of diameter ϵ . Therefore, for each $j \in \mathbb{N}$ there exists $S_j \subset S$ and invertible $L_j: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that, for all $x, y \in S_j$,

$$\|f(x) - f(y) - L_j(x - y)\| < \epsilon \|x - y\| \|L_j\|^{-1},$$

and $\mathcal{L}^n(S \setminus \bigcup_{j \in \mathbb{N}} S_j) = 0$.

Since $\mathcal{L}^n(S) > 0$, there exists $j \in \mathbb{N}$ with $\mathcal{L}^n(S_j) > 0$. Then S_j satisfies the hypotheses of *Lemma 9* and so $\mathcal{L}^n(\pi_V(S)) \geq \mathcal{L}^n(\pi_V(S_j)) > 0$ for all $V \in G(n, m)$ with $\|(\pi_V|_{L_j(\mathbb{R}^n)})^{-1}\|^{-1} \geq 2\epsilon$. Let $L_\epsilon = L_j$. Repeat this for each $i \in \mathbb{N}$ with $\epsilon = 1/i$. The set $G(n, m)$ is compact and so we may suppose that $L_{1/i}(\mathbb{R}^n) \rightarrow W \in G(n, m)$. The only $V \in G(n, m)$ for which $\mathcal{L}^n(\pi_V(S)) = 0$ satisfy $\|(\pi_V|_{L_{1/i}(\mathbb{R}^n)})^{-1}\|^{-1} < 2/i$ and hence $\|(\pi_V|_W)^{-1}\|^{-1} < 2/i$ for each $i \in \mathbb{N}$. That is, $V \cap W^\perp \neq \{0\}$ as required.

7.1 Exercises

Example 46

Let X be a metric space, $Y \subset X$ and $f: Y \rightarrow \mathbb{R}$ L -Lipschitz. Define $\tilde{f}: X \rightarrow \mathbb{R}$ by

$$\tilde{f}(x) = \sup\{f(y) - Ld(x, y) : y \in Y\}.$$

1. Show that \tilde{f} is an L -Lipschitz extension of f to X . This is called the *McShane–Whitney extension theorem*.
2. If $f: Y \rightarrow \mathbb{R}^n$ is L -Lipschitz, show that there is a $\sqrt{n}L$ -Lipschitz extension of f to X .
3. The following example shows that the vector valued extension cannot have the same Lipschitz constant in general:
Let

$$Y = \{(-1, 1), (1, -1), (1, 1)\} \subset \ell_\infty^2$$

and define

$$f(-1, 1) = (-1, 0), \quad f(1, -1) = (1, 0), \quad f(1, 1) = (0, \sqrt{3}).$$

Show that f is 1-Lipschitz but has no 1-Lipschitz extension to $Y \cup \{(0, 0)\}$.

4. However, the *Kirszbraun extension theorem* states that any Lipschitz map between any two Hilbert spaces may be extended whilst preserving the Lipschitz constant.
-

Example 47

1. Let $E \subset \mathbb{R}^m$ be n -rectifiable. Show that $\mathcal{H}^n|_E$ is σ -finite.
 2. Show that *Theorem 14* may not be true if E does not satisfy $\mathcal{H}^n(E) < \infty$.
-

PURELY UNRECTIFIABLE SETS

Definition 24 (Purely unrectifiable set)

A \mathcal{H}^n -measurable set $S \subset \mathbb{R}^m$ is purely n -unrectifiable if, for all n -rectifiable $E \subset \mathbb{R}^n$, $\mathcal{H}^n(S \cap E) = 0$.

Lemma 10

The four corner Cantor set $K \subset \mathbb{R}^2$ is purely 1-unrectifiable.*

Proof. Observe that the coordinate projections of K have \mathcal{L}^1 measure zero (see [Example 48](#)). If there existed a rectifiable $\gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(\gamma \cap K) > 0$, then $\gamma \cap K$ is a 1-rectifiable set of positive measure and hence, by [Corollary 2](#), one of the coordinate projections must have positive measure, a contradiction.

For a second proof see [Example 49](#).

Lemma 11

Let $A \subset \mathbb{R}^m$ be \mathcal{H}^m -measurable with $\mathcal{H}^m(A) < \infty$. There exists a decomposition $A = E \cup S$ with E n -rectifiable and S purely n -unrectifiable.*

Proof. Let

$$t = \sup\{\mathcal{H}^n(E) : E \subset A, n\text{-rectifiable}\}.$$

Since $\mathcal{H}^n(A) < \infty$, $t < \infty$. Let $E_i \subset A$ be n -rectifiable with $\mathcal{H}^n(E_i) \rightarrow t$. Then $E = \bigcup_{i \in \mathbb{N}} E_i$ is n -rectifiable and is contained in A . Therefore

$$t \geq \mathcal{H}^n(E) \geq \mathcal{H}^n(E_i) \rightarrow t$$

and so $\mathcal{H}^n(E) = t$. Then $S = A \setminus E$ is purely n -unrectifiable. Indeed, if $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $E' = (A \setminus E) \cap f(\mathbb{R}^n)$, is \mathcal{H}^m -measurable and so

$$t \geq \mathcal{H}^n(E \cup E') = \mathcal{H}^n(E) + \mathcal{H}^n(E') = t + \mathcal{H}^n(E').$$

We now state a very important theorem on the structure of purely unrectifiable sets. It requires the notion of a natural measure $\gamma_{n,m}$ on $G(n, m)$ that is invariant under the action of $SO(m)$. There are several ways to construct this measure.

The simplest is to consider $G(n, m)$ as a (compact) metric space equipped with the metric

$$d(V, W) = \|\pi_V - \pi_W\|.$$

Then $\gamma_{n,m}$ is given by (a scalar multiple of) $\mathcal{H}^{n(m-n)}$. We will not discuss the specific details of this measure. When $n = 1$, we may identify $G(1, m)$ with \mathbb{S}^{m-1} . In this case, $\gamma_{1,m}$ is simply \mathcal{H}^{m-1} .

Theorem 15 (Besicovitch–Federer projection theorem)

Let $S \subset \mathbb{R}^m$ be purely n -unrectifiable with $\mathcal{H}^n(S) < \infty$. Then, for $\gamma_{n,m}$ -a.e. $V \in G(n, m)$,

$$\mathcal{L}^n(\pi_V(S)) = 0.$$

Conversely, if $E \subset \mathbb{R}^m$ is purely n -unrectifiable with $\mathcal{H}^n(E) > 0$, for $\gamma_{n,m}$ -a.e. $V \in G(n, m)$,

$$\mathcal{L}^n(\pi_V(E)) > 0.$$

Remark 6

The converse statement is given by *Corollary 2*.

We will prove the projection theorem for $n = 1$ and $m = 2$, which was proved by Besicovitch. First we prove some preliminary geometric properties of purely unrectifiable sets.

Lemma 12

Let $E \subset \mathbb{R}^m$, $V \in G(m - n, m)$ and $0 < s < 1$. Suppose that, for every $x \in E$,

$$E \cap C(x, V, s) \cap B(x, r) = \emptyset.$$

Then E is n -rectifiable.

Proof. Since E may be divided into countably many sets of diameter at most r , we may suppose $\text{diam } E \leq r$. In this case, π_{V^\perp} has Lipschitz inverse on E . Indeed, if $x, y \in E$ then $y \notin C(x, V, s)$ and so

$$\|\pi_{V^\perp}(x - y)\| = \text{dist}(x - y, V) \geq s\|x - y\|.$$

Therefore, E is covered by a Lipschitz image of \mathbb{R}^n .

Lemma 13

Let $S \subset \mathbb{R}^m$ be purely n -unrectifiable, $V \in G(m - n, m)$, $0 < s < 1$ and $0 < \delta, \lambda < \infty$. If

$$\mathcal{H}^n(S \cap C(x, V, s) \cap B(x, r)) \leq \lambda r^n s^n \tag{8.1}$$

for every $x \in S$ and $0 < r < \delta$ then

$$\mathcal{H}^n(S \cap B(a, \delta/6)) \leq 2 \cdot 20^n \lambda \delta^n$$

for every $a \in S$.

Remark 7

Note that this is certainly not true for a rectifiable set E ; the first cone may be empty for every $x \in E$.

Proof. For a fixed $a \in S$, we may suppose $S \subset B(a, \delta/6)$. By *Lemma 12*, we may suppose that

$$S \cap C(x, V, s/4) \neq \emptyset$$

for every $x \in S$. For every $x \in S$ let

$$h(x) = \sup\{|x - y| : y \in S \cap C(x, V, s/4)\},$$

so that $0 < h(x) < \delta/3$ for all $x \in S$. Pick $x^* \in S \cap C(x, V, s/4)$ with $|x - x^*| \geq 3h(x)/4$ and let C_x be the cylinder

$$C_x = \pi_{V^\perp}^{-1}(\pi_{V^\perp}(B(x, sh(x)/4))).$$

We claim that

$$C_x \cap S \subset C(x, V, s) \cap B(x, 2h(x)) \cup C(x^*, V, s) \cap B(x^*, 2h(x)). \quad (8.2)$$

(Draw a picture!) Suppose $z \in C_x \cap S$ does not belong to the first set. Then

$$\begin{aligned} s\|x^* - z\| &\leq \|\pi_{V^\perp}(x^* - z)\| \\ &\leq \|\pi_{V^\perp}(x^* - x)\| + \|\pi_{V^\perp}(x - z)\| \\ &\leq s|x^* - x|/4 + sh(x)/4 \\ &\leq sh(x)/2, \end{aligned}$$

where the penultimate inequality follows because $x^* \in C(x, V, s/4)$ and $z \in C_x$. Therefore

$$\begin{aligned} \|x - z\| &\geq \|x - x^*\| - \|x^* - z\| \\ &> 3h(x)/4 - h(x)/2 \\ &\geq \|\pi_{V^\perp}(x - z)\|/s. \end{aligned}$$

That is, z belongs to the first set in (8.2).

By (8.2) and (8.1),

$$\mathcal{H}^1(S \cap C_x) \leq 2\lambda(2h(x)s)^n.$$

We apply *Lemma 4* to the balls

$$\pi_{V^\perp}(B(x, sh(x)/20))$$

with $x \in S$. This gives countably many $x_i \in S$, for which these balls are disjoint, and

$$S \subset \bigcup_{i \in \mathbb{N}} C_{x_i}.$$

Therefore

$$\begin{aligned} \mathcal{H}^n(S) &\leq \sum_{i \in \mathbb{N}} \mathcal{H}^n(C_{x_i}) \\ &\leq 2\lambda 2^n \sum_{i \in \mathbb{N}} (sh(x_i))^n \\ &= 2\lambda 2^n 20^n \sum_{i \in \mathbb{N}} \left(\frac{sh(x_i)}{20}\right)^n. \end{aligned}$$

But, the $\pi_{V^\perp}(B(x_i, sh(x_i)/20))$ are disjoint subsets of $B(\pi_{V^\perp}(a), \delta/2) \subset V^\perp$ and so the final sum is bounded above by $(\delta/2)^n$.

Corollary 3

If $S \subset \mathbb{R}^m$ is purely n -unrectifiable with $\mathcal{H}^n(S) < \infty$ then for every $V \in G(m - n, m)$, every $0 < s < 1$ and \mathcal{H}^n -a.e. $x \in S$,

$$\Theta^{*,n}(S \cap C(a, V, s), a) \geq 240^{-n-1} s^n.$$

Proof. For a fixed V, s , this is immediate from the fact that $\Theta^{*,n}(S, a) \geq 2^{-n}$ almost everywhere. To obtain the conclusion for all V, s , note that the conclusion is determined by a countable dense set of V, s .

Theorem 16

Let $E \subset \mathbb{R}^m$ satisfy $\mathcal{H}^n(E) < \infty$ and suppose that, for \mathcal{H}^n -a.e. $x \in E$, E has a unique approximate tangent plane at x . Then E is n -rectifiable.

Proof. By [Lemma 11](#), there exists a decomposition $E = E' \cup S$, where E' is n -rectifiable and S is purely n -unrectifiable. We must show that $\mathcal{H}^n(S) = 0$. Note that, by applying [Lemma 6](#) to E' , we see that the approximate tangent plane to E at $x \in S$ is also an approximate tangent plane to S for \mathcal{H}^n -a.e. x .

It suffices to show, for a fixed $W \in G(n, m)$, that the set S' of $x \in S$ whose approximate tangent plane V_x lies in $B(W, \delta)$ has measure zero. Suppose not. Then, for any $\lambda > 0$, there exists an $R > 0$ such that the set S'' of those $x \in S'$ with

$$\sup_{0 < r < R} \frac{\mathcal{H}^n(S \cap B(a, r) \setminus C(a, V_a, 1/3))}{r^n} < \lambda 3^{-n}$$

has positive measure. Fix an $x \in S''$. Since $\|\pi_{V_x} - \pi_W\| \leq 1/3$, for every $0 < r < R$ we have

$$C(x, W^\perp, 1/3) \cap B(x, r) \subset B(x, r) \setminus C(x, V_x, 1/3).$$

Thus, for $x \in S''$ and $0 < r < R$,

$$\mathcal{H}^n(S' \cap C(x, W^\perp, 1/3) \cap B(x, r)) < \lambda 3^{-n} r^n.$$

If $\lambda < 240^{-m-1}$, [Corollary 71](#) implies $\mathcal{H}^n(S'') = 0$, a contradiction.

For the remainder of the course, we will prove the Besicovitch projection theorem.

Theorem 17 (Besicovitch projection theorem)

Let $S \subset \mathbb{R}^2$ be purely 1-unrectifiable with $\mathcal{H}^1(S) < \infty$. Then for \mathcal{H}^1 -a.e. $e \in \mathbb{S}^1$,

$$\mathcal{L}^1(\pi_e(S)) = 0.$$

From [Corollary 3](#), we see that a purely unrectifiable set has many radiating out of almost every point in all directions at almost every point. We now precisely describe two ways in which this can occur.

Definition 25

Let $S \subset \mathbb{R}^2$ and $x \in S$. For $e \in \mathbb{S}^1$ let $l_e(x)$ be the half line $x + [0, \infty)e$ and for $I \subset \mathbb{S}^1$, let $C(I, x)$ be the cone $\bigcup_{e \in I} l_e(x)$. For $r > 0$ let $H_x(r)$ be those $e \in \mathbb{S}^1$ for which

$$|K \cap l_e(x) \cap B(x, r)| \geq 2.$$

That is, $S \cap l_e(x) \cap B(x, r)$ contains *another* point of K . Also let $H_x = \bigcap_{r>0} H_x(r)$, the directions that contain other points of S arbitrarily close to x . For $e \in \mathbb{S}^1$, we let H_e be those $x \in S$ for which $e \in H_x$.

For $R, M, \epsilon > 0$ let $D_x(R, M, \epsilon)$ be those $e \in \mathbb{S}^1$ for which there exists $0 < r < R$ and an interval $I \subset \mathbb{S}^1$ with $e \in I$ and $0 < \mathcal{H}^1(I) < \epsilon$ such that

$$\frac{\mathcal{H}^1(S \cap C(x, I) \cap B(x, r))}{r} \geq M\mathcal{H}^1(I).$$

That is, the *density* of S in the cone $C(x, I)$ at scale r is very high, compared to the length of I . Also let $D_x = \bigcap_{R, M, \epsilon > 0} D_x(R, M, \epsilon)$. For $e \in \mathbb{S}^1$, we let D_e be those $x \in S$ for which $e \in D_x$.

The main step in proving [Theorem 17](#) is the following.

Proposition 2

Let $S \subset \mathbb{R}^2$ be purely 1-unrectifiable with $\mathcal{H}^1(S) < \infty$. For \mathcal{H}^1 -a.e. $x \in S$, $\mathcal{H}^1(\mathbb{S} \setminus H_x \cup D_x) = 0$.

Before proving [Proposition 2](#), we will demonstrate how it is used to prove [Theorem 17](#).

Lemma 14 (Special case of the coarea formula)

For any $e \in \mathbb{S}^1$ and any compact $K \subset \mathbb{R}^2$,

$$\int_{\mathbb{R}} \text{card}(K \cap l_e(t)) dt \leq \mathcal{H}^1(K).$$

In particular, if $\mathcal{H}^1(K) < \infty$ then for any $e \in \mathbb{S}^1$, $\mathcal{L}^1(\pi_{e^\perp}(H_e)) = 0$.

Proof. Since K is compact,

$$f(t) = \text{card}(K \cap l_e(t))$$

is a Borel function. Indeed, if, for $\delta > 0$,

$$f_\delta(t) = \max\{n \in \mathbb{N} : \exists x_1, \dots, x_n \in K \cap l_e(t) \text{ with } \|x_i - x_j\| \geq \delta \forall 1 \leq i \neq j \leq n\}$$

then f_δ monotonically increases to f as $\delta \rightarrow 0$. Since K is compact, the f_δ are lower semi-continuous. Therefore, by the monotone convergence theorem, it suffices to bound the integral of each f_δ .

Fix $\delta > 0$ and cover K by sets E_1, E_2, \dots with $\text{diam } E_i < \delta$ such that

$$\sum_{i \in \mathbb{N}} \text{diam } E_i \leq \mathcal{H}^1(K) + \delta.$$

Note that

$$f_\delta(t) \leq \text{card}(\{i : E_i \cap l_e(t) \neq \emptyset\}).$$

Therefore

$$\begin{aligned}
 \int_{\mathbb{R}} f_{\delta} \, d\mathcal{L}^1 &\leq \int_{\mathbb{R}} \sum_{i \in \mathbb{N}} \chi_{\{(i,t): E_i \cap I_e(t) \neq \emptyset\}} \\
 &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \chi_{\{(i,t): E_i \cap I_e(t) \neq \emptyset\}} \\
 &\leq \sum_{i \in \mathbb{N}} \text{diam } E_i \\
 &\leq \mathcal{H}^1(K) + \delta,
 \end{aligned}$$

as required.

Lemma 15

Let $S \subset \mathbb{R}^2$ be \mathcal{H}^1 -measurable with $\mathcal{H}^1(S) < \infty$. Then for any $e \in \mathbb{S}^1$, $\mathcal{L}^1(\pi_{e^\perp}(D_e)) = 0$.*

Proof. Fix $e \in \mathbb{S}^1$. For any $M \in \mathbb{N}$ and $t \in \pi_{e^\perp}(D_e)$ there exists an $x \in D_e$, $r_x > 0$ and an interval $I \subset \mathbb{S}^1$ with $\text{diam } I < 1/10$ such that

$$\mathcal{H}^1(S \cap C(x, I) \cap B(x, r)) \geq M\mathcal{H}^1(rI).$$

Apply [Lemma 4](#) to the intervals $J_x = \pi_{e^\perp}(C(x, I) \cap B(x, r_x))$ to obtain a disjoint collection $J_{x_1}, J_{x_2}, \dots \subset \mathbb{R}$ such that

$$D_e \subset \bigcup_{i \in \mathbb{N}} 5J_{x_i}.$$

Therefore

$$\begin{aligned}
 \mathcal{L}^1(D_e) &\leq \sum_{i \in \mathbb{N}} 5\mathcal{L}^1(J_{x_i}) \\
 &\leq 5 \sum_{i \in \mathbb{N}} \mathcal{H}^1(rI_{x_i}) \\
 &\leq \frac{5}{M} \sum_{i \in \mathbb{N}} \mathcal{H}^1(S \cap C(x_i, I_{x_i}) \cap B(x_i, r_{x_i})) \\
 &\leq \frac{5}{M} \mathcal{H}^1(S),
 \end{aligned}$$

where the final inequality follows from the disjointness of the sets $C(x_i, I_{x_i}) \cap B(x_i, r_i)$, $i \in \mathbb{N}$. Since this is true for all $M \in \mathbb{N}$, $\mathcal{L}^1(D_e) = 0$.

Proof of [Theorem 17](#) using [Proposition 2](#).

Proof. By the inner regularity of \mathcal{H}^1 , it suffices to prove the result for compact S . By definition, we have

$$\{(x, e) \in S \times \mathbb{S}^1 : e \notin H_x \cup D_x\} = \{(x, e) : x \notin H_e \cup D_e\}.$$

[Proposition 2](#) implies that the left hand expression has $\mathcal{H}^1 \times \mathcal{H}^1$ -measure zero and so Fubini's theorem implies that, for \mathcal{H}^1 -a.e. $e \in \mathbb{S}^1$, $\mathcal{H}^1(S \setminus H_e \cup D_e) = 0$. Therefore, by [Lemma 15](#) and [Lemma 14](#), $\pi_{e^\perp}(S) = 0$ for \mathcal{H}^1 -a.e. $e \in \mathbb{S}^1$.

Proof of *Proposition 2*.

Proof. Fix $R, M, \epsilon > 0$ and $x \in S$ which satisfies the conclusion of *Corollary 3*. That is,

$$\Theta^{*,1}(S \cap C(x, I), x) \geq c_0 \mathcal{H}^1(I) \tag{8.3}$$

for every interval $I \subset \mathbb{S}^1$. It suffices to show that $\mathcal{H}^1(\mathbb{S}^1 \setminus H_x(R) \cup D_x(R, M, \epsilon)) = 0$. In fact, we will show that, for any $e \in \mathbb{S}^1$,

$$\Theta^{*,1}(H_x(R), e) > 0 \quad \text{or} \quad \Theta_*^1(D_x(R, M, \epsilon), e) > 0,$$

from which the result follows by the Lebesgue density theorem.

To this end, fix $e \in \mathbb{S}^1$ with $\Theta^{*,1}(H_x(R), e) = 0$. Then for all sufficiently small intervals I with $e \in I \subset \mathbb{S}^1$,

$$\mathcal{H}^1(H_x(R) \cap I) < c_0 \mathcal{H}^1(I)/4M. \tag{8.4}$$

Fix such an I . By (8.3), there exists $r < R$ with

$$\mathcal{H}^1(S \cap C(x, I) \cap B(x, r)) \geq c_0 r \mathcal{H}^1(I). \tag{8.5}$$

Note that (8.4) implies

$$\mathcal{H}^1(H_x(r) \cap I) < c_0 \mathcal{H}^1(I)/4M. \tag{8.6}$$

We will prove that

$$\mathcal{H}^1(D_x(R, M, \epsilon) \cap I) \geq c_0 \mathcal{H}^1(I)/4M. \tag{8.7}$$

Since I is any sufficiently small interval containing e , this implies

$$\Theta_*^1(D_x(R, M, \epsilon), e) \geq c_0/4M > 0$$

as required.

By (8.6) we may cover $H_x(r) \cap I$ by disjoint intervals I_1, I_2, \dots with

$$\sum_{i \in \mathbb{N}} \mathcal{H}^1(I_i) < c_0 \mathcal{H}^1(I)/4M$$

(indeed, the disjointness of the intervals is shown in *Example 53*). By the definition of $H_x(r)$, we know that

$$S \cap C(x, I) \cap B(x, r) \subset \bigcup_{i \in \mathbb{N}} S \cap C(x, I_i) \cap B(x, r). \tag{8.8}$$

Let \mathcal{G} be those $i \in \mathbb{N}$ with

$$\frac{\mathcal{H}^1(S \cap C(x, I_i) \cap B(x, r))}{r} \geq M \mathcal{H}^1(I_i). \tag{8.9}$$

Note that by (8.5) and (8.8),

$$\begin{aligned} \sum_{i \in \mathcal{G}} \frac{\mathcal{H}^1(S \cap C(x, I_i) \cap B(x, r))}{r} &\geq \frac{\mathcal{H}^1(S \cap C(x, I) \cap B(x, r))}{r} - \frac{c_0 \mathcal{H}^1(I)}{4} \\ &\geq \frac{3c_0 \mathcal{H}^1(I)}{4}. \end{aligned}$$

That is, the cones associated to $G := \bigcup_{i \in \mathcal{G}} I_i$ cover a large proportion of $S \cap C(x, I) \cap B(x, r)$. Moreover, $G \subset D_x(R, M, \epsilon)$, because if $\xi \in G$ then $\xi \in I_i$ for some $i \in \mathcal{G}$ which satisfies the definition of $D_x(R, M, \epsilon)$.

Thus, to show (8.7), it would be enough to bound $\mathcal{H}^1(G)$ from below by a multiple of $\mathcal{H}^1(I)$. But this is not necessarily true: the intervals that form G could be extremely thin compared to I . To accommodate this, we enlarge the intervals I_i with $i \in \mathcal{G}$ as follows. For each $i \in \mathcal{G}$ enlarge I_i until (8.9) becomes an equality or until I_i intersects another I_j . If the first possibility occurs then we still have $I_i \subset D_x(R, M, \epsilon)$. If the second possibility occurs then we merge the two intervals; since both sides of (8.9) are linear in I , and the boundary of each $C(x, I_i)$ contains no points of S , (8.9) remains true after the merge. By the same reasoning, both sides of (8.9) are continuous under expanding I and consequently one of the two possibilities must occur.

This results in a disjoint collection of intervals \tilde{I}_i for which (8.9) is an equality. Moreover,

$$G = \bigcup_{i \in \mathcal{G}} I_i \subset \bigcup \tilde{I}_i$$

and, by construction, each $\tilde{I}_i \in D_x(R, M, \epsilon)$. Therefore

$$\begin{aligned} \mathcal{H}^1(D_x(r_0, \epsilon, M) \cap I) &\geq \sum \mathcal{H}^1(\tilde{I}_i) \\ &= \sum \frac{\mathcal{H}^1(S \cap B(x, r) \cap C(x, \tilde{I}_i))}{rM} \\ &\geq \sum_{i \in \mathcal{G}} \frac{\mathcal{H}^1(S \cap B(x, r) \cap C(x, I_i))}{rM} \\ &\geq \frac{3c_0 \mathcal{H}^1(I)}{4M}. \end{aligned}$$

8.1 Exercises

Example 48

Prove that the coordinate projections of the four corner Cantor set $K \subset \mathbb{R}^2$ have Lebesgue measure zero.

Example 49

A second proof that the four corner Cantor set is purely 1 unrectifiable.

Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}^2$ satisfies $\mathcal{H}^1(f(\mathbb{R}) \cap K) > 0$.

1. Prove that there exists $x \in \mathbb{R}$ that is a density point of $f^{-1}(K)$ such that $f'(x) \neq 0$.
 2. Therefore, for sufficiently small r , $f(x - r, x + r)$ is approximated by a line segment of length $2r|f'(x)|$ that is mostly contained in K . Derive a contradiction.
-

Example 50

Prove that the decomposition given in [Lemma 11](#) is unique up to \mathcal{H}^n -null sets.

Example 51

Think about *Corollary 3* and *Proposition 2* in regard to the four corner Cantor set.

Example 52

Let $K \subset \mathbb{R}^2$ be compact. Show that

$$\{(x, e) : e \in H_x\} \quad \text{and} \quad \{(x, e) : e \in D_x\}$$

are Borel subsets of $K \times \mathbb{S}^1$.

Example 53

Show that in the definitions of \mathcal{L}^n and \mathcal{H}^n we may suppose the covering intervals, respectively sets, may be chosen to be disjoint.

Example 54 (Open problem)

For $k \geq 2$, does there exist a compact purely 1-unrectifiable $S \subset \mathbb{R}^2$ with $\mathcal{H}^1(S) > 0$ that intersects every line in at most k points?

SOME STANDARD THEOREMS

Theorem 18 (Egorov's theorem)

Let μ be a finite measure on a set X and $f_n : X \rightarrow \mathbb{R}$ a sequence of μ -measurable functions such that $f_n \rightarrow f$ pointwise μ -a.e. Then for every $\epsilon > 0$ there exists a measurable $G \subset X$ with $\mu(X \setminus G) < \epsilon$ and $f_n \rightarrow f$ uniformly on G .

Proof. Fix $\epsilon > 0$, $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$ let

$$B_{n,k} = \{x \in X : |f_n(x) - f(x)| > 1/k \text{ for some } m \geq n\}.$$

By assumption, the $B_{n,k}$ are measurable sets that monotonically decrease to a μ -null set as $n \rightarrow \infty$. Therefore, there exists $n \in \mathbb{N}$ such that $\mu(B_{n,k}) < \epsilon 2^{-k}$. Let

$$G_k = X \setminus B_{n,k} \quad \text{and} \quad G = \bigcap_{k \in \mathbb{N}} G_k.$$

Then $\mu(X \setminus G) < \epsilon$ and, for each $x \in G$ and each $k \in \mathbb{N}$, $G \subset G_k$, so there exists $n \in \mathbb{N}$ such that

$$|f(x) - f_m(x)| < 1/k$$

for all $m \geq n$. That is, $f_m \rightarrow f$ uniformly on G , as required.

Theorem 19 (Lusin's theorem)

Let μ be a finite Borel measure on a metric space X and let $f : X \rightarrow \mathbb{R}$ be μ -measurable. Then for every $\epsilon > 0$ there exists a closed $C \subset X$ with $\mu(X \setminus C) < \epsilon$ such that $f|_C$ is continuous.

Proof. Fix $\epsilon > 0$ and for each $i \in \mathbb{Z}$ let

$$X_i = f^{-1}([i\epsilon, (i+1)\epsilon)),$$

a collection of disjoint Borel sets which cover X . Since $\mu(X) < \infty$, there exists $n \in \mathbb{N}$ such that

$$\mu\left(X \setminus \bigcup_{i=1}^n X_i\right) < \epsilon.$$

Since μ is Borel, for each $1 \leq i \leq n$ there exists a closed $C_i \subset X_i$ with $\mu(X_i \setminus C_i) < \epsilon/n$.

For a moment fix $1 \leq i \leq n$ and let

$$D = \bigcup_{1 \leq j \neq i \leq n} C_j.$$

Since D is closed, the sets $B(D, \delta)$ monotonically decrease to D as $\delta \rightarrow 0$, which is disjoint from C_i . Therefore there exists $\delta_i > 0$ such that

$$C'_i := C_i \setminus B(D, \delta_i)$$

satisfies $\mu(C_i \setminus C'_i) < \epsilon/n$. Note that C'_i is closed.

Let $\delta := \min_{1 \leq i \leq n} \delta_i > 0$ and $C_\epsilon = \bigcup_{i=1}^n C'_i$, a closed set. For any $1 \leq i \neq j \leq n$ and $x \in C'_i$ and $y \in C'_j$, $d(x, y) \geq \delta$. Therefore, if $x, y \in C_\epsilon$ with $d(x, y) < \delta$, $|f(x) - f(y)| < \epsilon$. Repeat this for each $k \in \mathbb{N}$ with $\epsilon_k = 2^{-k}\epsilon/3$ and let $C = \bigcap_{k \in \mathbb{N}} C_{\epsilon_k}$, so that $\mu(X \setminus C) < \epsilon$. Then f is continuous on C .

Definition 26 (Absolutely continuous and singular measures)

Let μ, ν be measures on a set X . We say ν is absolutely continuous with respect to μ , written $\nu \ll \mu$, if for every $S \subset X$, $\mu(S) = 0 \implies \nu(S) = 0$. We say that ν is singular with respect to μ , written $\nu \perp \mu$ if there exists $A \subset X$ with $\mu(X \setminus A) = 0 = \nu(A)$.

Let μ be a measure on a set X and let $f: X \rightarrow \mathbb{R}^+$. The set valued function

$$\nu(S) = \int_S f \, d\mu$$

defines a measure on X . Note also that $\nu \ll \mu$. The Radon-Nikodym theorem provides the converse to this statement.

Theorem 20 (Radon–Nikodym theorem)

Let μ, ν be finite measures on a set X such that $\nu \ll \mu$. There exists a μ and ν -measurable $f: X \rightarrow \mathbb{R}^+$ such that

$$\nu(S) = \int_S f \, d\mu$$

for all μ and ν -measurable $S \subset X$. The function f is called the Radon-Nikodym derivative of ν with respect to μ .

Proof. Let \mathcal{F} be the set of all μ and ν -measurable $f: X \rightarrow \mathbb{R}^+$ such that

$$\int_S f \, d\mu \leq \nu(S)$$

for all μ and ν -measurable $S \subset X$. Note that $0 \in \mathcal{F}$ and

$$f, g \in \mathcal{F} \implies \max\{f, g\} \in \mathcal{F}. \tag{9.1}$$

Let

$$M = \sup \left\{ \int f \, d\mu : f \in \mathcal{F} \right\},$$

so that $0 \leq M \leq \nu(X) < \infty$, and let $f_i \in \mathcal{F}$ be such that

$$\int f_i \, d\mu \rightarrow M.$$

(9.1) implies that we may suppose the f_i monotonically increase. Let $f: X \rightarrow \mathbb{R}^+$ be the pointwise limit of the f_i . Then f is μ and ν -measurable and, by the monotone convergence theorem, $f \in \mathcal{F}$ and $\int f \, d\mu \geq M$. Thus

$$\int f \, d\mu = M.$$

We claim that f satisfies the conclusion of the proposition. Indeed, suppose that $B \subset X$ is μ -measurable but

$$\nu(B) > \int_B f \, d\mu$$

and let $\epsilon > 0$ be such that

$$\nu(B) > \int_B f + \epsilon \, d\mu. \tag{9.2}$$

Let \mathcal{S} be the collection of all μ and ν -measurable $S \subset B$ such that

$$\nu(S) \leq \int_S f + \epsilon \, d\mu.$$

We claim that there exists a μ and ν -measurable $G \subset B$ of positive μ -measure such that each μ and ν -measurable $G' \subset G$ of positive μ -measure is not contained in \mathcal{S} . Indeed, if not, then each $G \subset B$ of positive μ -measure contains an element of \mathcal{S} of positive μ -measure. Thus *Lemma 15* gives a countable disjoint decomposition of μ -almost all of B into elements of \mathcal{S} . Since $\nu \ll \mu$ this implies

$$\nu(B) \leq \int_B f + \epsilon \, d\mu,$$

contradicting (9.2).

Note that $f + \epsilon G \in \mathcal{F}$. Indeed, if $S \subset X$ is μ -measurable,

$$\begin{aligned} \int_S f + \epsilon G \, d\mu &= \int_{S \setminus G} f + \epsilon G \, d\mu + \int_{S \cap G} f + \epsilon G \, d\mu \\ &\leq \nu(S \setminus G) + \nu(S \cap G) \\ &= \nu(S). \end{aligned}$$

On the other hand, since $\mu(G) > 0$,

$$\int f + \epsilon G \, d\mu = M + \epsilon \mu(G) > M,$$

contradicting the definition of M .

Theorem 21 (Lebesgue decomposition theorem)

Let μ, ν be finite measures on a set X . There exists a ν -measurable $A \subset X$ with $\mu(X \setminus A) = 0$ such that, for all $S \subset A$, $\mu(S) = 0 \Rightarrow \nu(S) = 0$. That is, $\nu = \nu_{ac} + \nu_{\perp}$ with $\nu_{ac} \ll \mu$ and $\nu_{\perp} \perp \mu$.

Proof. Let \mathcal{S} be the set of all ν -measurable $S \subset X$ with $\mu(S) = 0$. By *Lemma 15*, there exists $S_i \in \mathcal{S}$ such that each $S \in \mathcal{S}$ with

$$S \subset A := X \setminus \bigcup_{i \in \mathbb{N}} S_i$$

satisfies $\nu(S) = 0$. Since $\mu(X \setminus A) = 0$, this is the required decomposition.

9.1 Exercises

Example 55

Let μ be a Borel measure on a metric space X , $f: X \rightarrow \mathbb{R}$ μ -integrable and $\epsilon > 0$.

1. Show that there exists a simple function s such that

$$\int_X |f - s| d\mu < \epsilon.$$

2. If f is positive show that we may require $0 \leq s \leq f$ in the previous point.
3. Show that if μ is finite, there exists $g \in C(X)$ with

$$\int_X |f - g| d\mu < \epsilon.$$

4. Show that the previous point may fail if μ is only σ -finite.
-

Example 56

Prove the following variant of Lusin's theorem for the case that μ is not finite but f is μ -integrable: for every $\epsilon > 0$ there exists a closed $C \subset X$ with

$$\int_{X \setminus C} |f| d\mu < \epsilon$$

such that $f|_C$ is continuous.

Example 57

Let μ, ν be measures on a set X . Show that if $A \subset X$ is $\mu + \nu$ measurable then it is also μ -measurable.

Example 58

Prove the Radon-Nikodym and Lebesgue decomposition theorems for σ -finite measures.

Example 59

Let μ, ν be finite measures on a set X and suppose $\nu \ll \mu$. For any μ, ν -measurable $g: X \rightarrow \mathbb{R}^+$ show that

$$\int g d\nu = \int gf d\mu,$$

where f is the Radon-Nikodym derivative of ν with respect to μ .

Example 60

For a measure μ on a set X , we say a function $f: X \rightarrow \mathbb{R}^n$ is μ -measurable if each component of f is μ -measurable and define the integral of f component by component.

An n -dimensional vector valued measure is a function

$$\nu: \{A : A \subset X\} \rightarrow \mathbb{R}^n$$

for which there exists a measure μ on X and a μ -measurable $f: X \rightarrow \mathbb{S}^{n-1}$ such that

$$\nu(A) = \int_A f \, d\mu$$

for each $A \subset X$.

1. Show that if

$$\nu = \int f \, d\mu = \int f' \, d\mu'$$

are two representations of a vector valued measure then $\mu = \mu'$ (when restricted to the set of μ -measurable sets), and hence $f = f'$ μ -a.e. We denote this unique measure by $|\nu|$. It is called the total variation of ν .

2. Show that the set of all vector valued measures form a normed vector space when equipped with

$$\|\nu\| = |\nu|(X).$$

3. Show that this space is complete.

A signed measure is a 1-dimensional vector valued measure.

Example 61

Let Σ be a σ -algebra on X . The standard definition of a signed measure on Σ is a countably additive function

$$\mu: \Sigma \rightarrow \mathbb{R}.$$

The *Hahn decomposition theorem* states that there exist disjoint $P, N \in \Sigma$ with $X = P \cup N$ such that:

- For every $S \in \Sigma$ with $S \subset P$, $\mu(S) \geq 0$ and
- For every $S \in \Sigma$ with $S \subset N$, $\mu(S) \leq 0$.

That is, $\mu|_P$ and $-\mu|_N$ are (positive) measures.

Use the Lebesgue decomposition and Radon–Nikodym theorems to show that the two definitions of a signed measure agree.

FUBINI'S THEOREM

Definition 27

Let μ, ν be measures on sets X, Y respectively and define the set of *rectangles* to be

$$\mathcal{R} = \{A \subset X : A \text{ } \mu\text{-measurable}\} \otimes \{B \subset Y : B \text{ } \nu\text{-measurable}\}.$$

The *product measure* $\mu \times \nu$ on $X \times Y$ is defined by

$$\mu \times \nu(S) = \inf \sum_{i \in \mathbb{N}} \mu(A_i) \nu(B_i),$$

where the infimum is taken over all countable collections of rectangles $A_i \times B_i \in \mathcal{R}$ with

$$S \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i.$$

This is a measure, see [Example 62](#).

Lemma 16

Let X, Y be sets and μ, ν measures on X, Y respectively. Then $\mu \times \nu$ is equivalently defined by the formula

$$\mu \times \nu(S) = \inf \sum_{i \in \mathbb{N}} \mu(A_i) \nu(B_i),$$

where the infimum is taken over all disjoint countable collections of rectangles $A_i \times B_i \in \mathcal{R}$ with

$$S \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i.$$

Proof. If A, C are μ -measurable and B, D are ν -measurable then

$$\begin{aligned} (A \times B) \setminus (C \times D) &= [(A \setminus C) \times B] \cup [(A \cap C) \times (B \setminus D)] \\ &:= A_1 \times B_1 \cup A_2 \times B_2 \end{aligned}$$

is a decomposition into disjoint rectangles. Since A, C and B, D are μ and ν measurable respectively,

$$\begin{aligned} \mu(A_1) \nu(B_1) + \mu(A_2) \nu(B_2) &= [\mu(A) - \mu(A \cap C)] \nu(B) + \mu(A \cap C) [\nu(B) - \nu(B \cap D)] \\ &= \mu(A) \nu(B) - \mu(A \cap C) \nu(B \cap D) \\ &\leq \mu(A) \nu(B). \end{aligned}$$

Thus the two formulae agree.

Theorem 22 (Fubini's theorem)

Let X, Y be sets and μ, ν σ -finite measures on X, Y respectively.

1. If A is μ -measurable and B is ν -measurable then $A \times B$ is $\mu \times \nu$ -measurable and

$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$

2. If S is $\mu \times \nu$ -measurable then

$$S^y := \{x \in X : (x, y) \in S\}$$

is μ -measurable for ν -a.e. $y \in Y$, $y \mapsto \mu(S^y)$ is ν -measurable;

$$S_x := \{y \in Y : (x, y) \in S\}$$

is ν -measurable for μ -a.e. $x \in X$, $x \mapsto \nu(S_x)$ is μ -measurable; and

$$\mu \times \nu(S) = \int_Y \mu(S^y) d\nu(y) = \int_X \nu(S_x) d\mu(x).$$

3. If $f: X \times Y \rightarrow \mathbb{R}^+$ is $\mu \times \nu$ -measurable or $f: X \times Y \rightarrow \mathbb{R}$ is $\mu \times \nu$ integrable then

$$\int_{X \times Y} f d\mu \times \nu = \int_X \int_Y f d\nu(y) d\mu(x) = \int_Y \int_X f d\mu(x) d\nu(y).$$

Proof. Note that the third point follows from the second by the monotone convergence theorem. We prove the theorem for finite μ, ν .

To begin, let

$$\mathcal{U} := \left\{ \bigcup_{i \in \mathbb{N}} R_i : R_i \in \mathcal{R} \text{ pairwise disjoint} \right\}.$$

Let \mathcal{P} be the set of $S \subset X \times Y$ such that $y \mapsto \mu(S^y)$ is ν -measurable and for any $S \in \mathcal{P}$ define

$$\rho(S) := \int_Y \mu(S^y) d\nu.$$

Observe that $\rho(S)$ is monotonic in S .

If $S = A \times B \in \mathcal{R}$ then $y \mapsto \mu(S^y) = \mu(A)\chi_B$ is ν -measurable and $\rho(S) = \mu(A)\nu(B)$. If $U \in \mathcal{U}$ with

$$U = \bigcup_{i \in \mathbb{N}} A_i \times B_i$$

a disjoint union, then

$$y \mapsto \mu(U^y) = \sum_{i \in \mathbb{N}} \mu(A_i)\chi_{B_i}$$

is a countable sum of ν -measurable functions and so $U \in \mathcal{P}$. Moreover,

$$\rho(U) = \int_Y \mu(U^y) d\nu(y) = \sum_{i \in \mathbb{N}} \mu(A_i) \nu(B_i). \quad (10.1)$$

Thus, for any $S \subset X \times Y$,

$$\mu \times \nu(S) = \inf\{\rho(U) : S \subset U \in \mathcal{U}\}. \quad (10.2)$$

To prove the second conclusion, let A be μ -measurable and B be ν -measurable. By definition, $\rho(A \times B) = \mu(A)\nu(B)$ and, since ρ is monotonic, $\rho(A \times B) \leq \rho(U)$ whenever $A \times B \subset U \in \mathcal{U}$. Thus, by (10.2),

$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$

Let $E \subset X \times Y$ and $E \subset U \in \mathcal{U}$. Observe

$$U \cap (A \times B) \quad \text{and} \quad U \setminus (A \times B)$$

are disjoint members of \mathcal{U} . Therefore, by (10.1) and (10.2),

$$\begin{aligned} \rho(U) &= \rho(U \cap (A \times B)) + \rho(U \setminus (A \times B)) \\ &\geq \mu \times \nu(E \cap (A \times B)) + \mu \times \nu(E \setminus (A \times B)). \end{aligned}$$

When taking the infimum over all such U , the left hand side converges to $\mu \times \nu(E)$, and so $A \times B$ is $\mu \times \nu$ -measurable. This also implies that all elements of \mathcal{U} are measurable.

Let $S \subset X \times Y$ and suppose that $U_1, U_2, \dots \in \mathcal{U}$ are such that $\rho(U_i) \rightarrow \mu \times \nu(S)$. Since the intersection of any two rectangles is a rectangle,

$$V_i := U_i \cap U_{i-1} \cap \dots \cap U_1 \in \mathcal{U}$$

for each $i \in \mathbb{N}$. Moreover, $\rho(V_i)$ monotonically decreases to $\mu \times \nu(S)$. Let $W = \bigcap_{i \in \mathbb{N}} V_i \supset S$. Note that, for each $y \in Y$, $\mu(V_i^y)$ monotonically decreases to $\mu(W^y)$, so that $W \in \mathcal{P}$. Since μ, ν are finite, the monotone convergence theorem implies that $\rho(V_i) \rightarrow \rho(W)$ and hence $\mu \times \nu(S) = \rho(W)$. Since $\mu \times \nu(W) \geq \mu \times \nu(S)$, the monotonicity of ρ implies $\mu \times \nu(W) = \rho(W)$.

To prove the first conclusion, if S is $\mu \times \nu$ -measurable, then $\mu \times \nu(W \setminus S) = 0$. By (10.2) there exists $Z \supset W \setminus S$ with $\rho(Z) = 0$. That is, $\mu(W^y) = \mu(Z^y)$ for ν -a.e. $y \in Y$, and hence the first conclusion of the second point. Moreover, $\rho(S) = \rho(W) = \mu \times \nu(S)$, which concludes the proof.

10.1 Exercises

Example 62

Let X, Y be sets and μ, ν measures on X, Y respectively. Prove that $\mu \times \nu$ is a measure on $X \times Y$.

Example 63

Let X, Y be separable metric spaces. Show that

$$\mathcal{B}(X \times Y) = \Sigma(\mathcal{B}(X) \otimes \mathcal{B}(Y)).$$

Example 64

Let X, Y be separable metric spaces and μ, ν finite Borel measures on X, Y respectively. Show that $\mu \times \nu|_{\mathcal{B}(X \times Y)}$ is the unique countably additive set function on $\mathcal{B}(X \times Y)$ satisfying $\mu(A \times B) = \mu(A) \times \nu(B)$ for all $A \times B \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$.

Example 65

Prove Fubini's theorem, *Example 63* and *Example 64* for σ -finite measures μ, ν .

Example 66

Show that the third conclusion of *Theorem 22* may fail if

1. $f: X \times Y \rightarrow \mathbb{R}$ is measurable but not integrable; Hint: consider μ, ν the counting measure on \mathbb{N} . Exploit the "identity"

$$(1 - 1) + (1 - 1) + \dots = 1 + (-1 + 1) + (-1 + 1) + \dots$$

2. $f: X \times Y \rightarrow \mathbb{R}^+$ is measurable but μ is not σ -finite. Hint: consider $\mu = \mathcal{L}^1, \nu$ the counting measure on \mathbb{R} .

Prove the third conclusion of *Theorem 22* for $f: X \times Y \rightarrow \mathbb{R}$ integrable, even if μ, ν are not σ -finite.

Example 67

Note in the second conclusion of *Theorem 22*, we must exclude a set of measure zero. Indeed, if \mathcal{V} is a Vitali set, note that $\mathcal{V} \times \{0\} \subset \mathbb{R}^2$ is \mathcal{L}^2 -measurable.

Example 68

Let X be a separable metric space and $f: X \rightarrow [0, \infty)$ a Borel function. Prove that

$$\int_X f \, d\mu = \int_0^\infty \mu(\{x \in X : f(x) \geq t\}) \, dt.$$

Hint: consider

$$A = \{(x, t) : f(x) \geq t\}.$$

Example 69

A measure μ on a metric space X is *uniformly distributed* if there exists a $g: (0, \infty) \rightarrow (0, \infty)$ such that $\nu(B(x, r)) = g(r)$ for all $x \in X$ and $r > 0$. Let μ, ν be uniformly distributed Borel regular measures on a separable metric space X (with functions g and h respectively). Let $U \subset X$ be open.

1. Observe that, for any $x \in U$,

$$\lim_{r \rightarrow 0} \frac{\nu(U \cap B(x, r))}{h(r)} = 1$$

for every $x \in U$.

2. Deduce that

$$\mu(U) \leq \liminf_{r \rightarrow 0} h(r)^{-1} \int_U \nu(U \cap B(x, r)) \, d\mu(x).$$

3. Deduce that

$$\mu(U) \leq \liminf_{r \rightarrow 0} h(r)^{-1} \int_U \mu(U \cap B(y, r)) \, d\nu(y) = \liminf_{r \rightarrow 0} \frac{g(r)}{h(r)} \nu(U).$$

Deduce that $\mu = c\nu$ for some $c > 0$.

INDEX

- genindex
- search

5r

5r (*covering*), 25

HL_not_bounded

HL_not_bounded (*covering*), 30

abs-cont-fun

abs-cont-fun (*rademacher*), 31

almost-everywhere

almost-everywhere (*measures*), 8

alt_meas_f

alt_meas_f (*integration*), 16

approximate-tangent

approximate-tangent (*rectifiable*), 41

banach-alaoglu

banach-alaoglu (*daniell*), 24

basic-rect-crit

basic-rect-crit (*purely_unrectifiable*), 48

besicovitch

besicovitch (*purely_unrectifiable*), 50

besicovitch-main

besicovitch-main (*purely_unrectifiable*), 51

borel-is-regular

borel-is-regular (*measures*), 11

borel-product

borel-product (*fubini*), 65

bv-abs-lip

bv-abs-lip (*rademacher*), 33

bv-inc

bv-inc (*rademacher*), 33

cantor-projections

cantor-projections (*purely_unrectifiable*), 54

cantor-pu

cantor-pu (*purely_unrectifiable*), 54

cantor-set

cantor-set (*measures*), 10

caradtheodory-construction

caradtheodory-construction (*measures*), 7

caradtheodory-criterion

caradtheodory-criterion (*measures*), 6

coarea

coarea (*purely_unrectifiable*), 51

converse-tangent-plane

converse-tangent-plane (*purely_unrectifiable*), 50

daniell

daniell (*daniell*), 23

daniell_not_unique

daniell_not_unique (*daniell*), 23

daniell_unique

daniell_unique (*daniell*), 20

def-sigm-algebra

def-sigm-algebra (*measures*), 5

def:measurable-set

def:measurable-set (*measures*), 3

def:measure

def:measure (*measures*), 3

definition-0

definition-0 (*hausdorff_measure*), 37

definition-10

definition-10 (*measures*), 7

definition-11

definition-11 (*purely_unrectifiable*), 50

definition-12

definition-12 (*measures*), 8

definition-2

definition-2 (*standard_theorems*), 58

definition-3

definition-3 (*measures*), 3

definition-6

definition-6 (*covering*), 27

densities

densities (*hausdorff_measure*), 37

density-projection

density-projection (*purely_unrectifiable*), 52

dominated-convergence

dominated-convergence (*integration*), 14

doubling-measure

doubling-measure (*covering*), 26

egorov

egorov (*standard_theorems*), 57

ex-maximal-balls

ex-maximal-balls (*covering*), 29

example-10

example-10 (*rademacher*), 34

example-11

example-11 (*standard_theorems*), 61

example-12

example-12 (*integration*), 16

example-13

example-13 (*integration*), 16

example-17

example-17 (*purely_unrectifiable*), 54

example-18

example-18 (*purely_unrectifiable*), 54

example-19

example-19 (*purely_unrectifiable*), 55

example-21

example-21 (*purely_unrectifiable*), 55

example-25

example-25 (*measures*), 11

example-6

example-6 (*standard_theorems*), 60

example-7

example-7 (*rademacher*), 33

example-8

example-8 (*standard_theorems*), 60

example-9

example-9 (*standard_theorems*), 60

exhaustion

exhaustion (*measures*), 8

ext_meas_not_unique

ext_meas_not_unique (*measures*), 11

extend-outer-meas

extend-outer-meas (*measures*), 9

fatou

fatou (*integration*), 13

fubini

fubini (*fubini*), 64

fubini-prelim

fubini-prelim (*fubini*), 63

hardly-littlewood

hardly-littlewood (*covering*), 27

hausdorff-defn

hausdorff-defn (*measures*), 10

hd-constant-lebesguehd-constant-lebesgue (*fubini*), 66**hd-constant-lebesgue-2**hd-constant-lebesgue-2 (*covering*), 30**hd-disjoint**hd-disjoint (*purely_unrectifiable*), 55**hd-separable**hd-separable (*hausdorff_measure*), 40**hs-borel**hs-borel (*hausdorff_measure*), 40**integral**integral (*integration*), 13**l1-dense**l1-dense (*standard_theorems*), 60**lattice**lattice (*daniell*), 17**lebesgue-decomposition**lebesgue-decomposition (*standard_theorems*), 59**lebesgue-defn**lebesgue-defn (*measures*), 8**lebesgue-density**lebesgue-density (*covering*), 29**lebesgue-diff**lebesgue-diff (*covering*), 28**lemma-1**lemma-1 (*purely_unrectifiable*), 47**lemma-4**lemma-4 (*measures*), 3**limits-of-measurable**limits-of-measurable (*integration*), 15**lower-density**lower-density (*hausdorff_measure*), 39**lusin**lusin (*standard_theorems*), 57**many-directions**many-directions (*purely_unrectifiable*), 48**maximal-function-borel**maximal-function-borel (*covering*), 29**measurable-for-sum**measurable-for-sum (*standard_theorems*), 60**measurable-function**measurable-function (*integration*), 13**measurable-properties**measurable-properties (*measures*), 4**monotone-convergence**monotone-convergence (*integration*), 14**monotone_daniell**monotone_daniell (*daniell*), 18**must-use-c0**must-use-c0 (*daniell*), 23**pre-tangent**pre-tangent (*rectifiable*), 42**pre-vitali**pre-vitali (*covering*), 25**product-measure**product-measure (*fubini*), 63**product-measure-is-a-measure**product-measure-is-a-measure (*fubini*), 65**proj-of-rect**proj-of-rect (*rectifiable*), 43**projection-of-rectifiable-cor**projection-of-rectifiable-cor (*rectifiable*), 44**ptwise_unif**ptwise_unif (*daniell*), 23**pu-density-cones**pu-density-cones (*purely_unrectifiable*), 50**purely-unrectifiable**purely-unrectifiable (*purely_unrectifiable*), 47

radon-nikodym

radon-nikodym (*standard_theorems*), 58

rect-decomp

rect-decomp (*purely_unrectifiable*), 47

rectifiable-approx-tangent

rectifiable-approx-tangent (*rectifiable*), 43

rectifiable-set

rectifiable-set (*rectifiable*), 41

remark-2

remark-2 (*measures*), 3

remark-4

remark-4 (*purely_unrectifiable*), 48

remark-7

remark-7 (*rectifiable*), 44

remark-9

remark-9 (*measures*), 7

restriction

restriction (*measures*), 10

riesz

riesz (*daniell*), 22

riesz_obtain_borel

riesz_obtain_borel (*daniell*), 20

sard

sard (*rectifiable*), 41

sigma-alg-generated

sigma-alg-generated (*measures*), 9

simple-function

simple-function (*integration*), 13

simple-integral-properties

simple-integral-properties (*integration*), 15

theorem-2

theorem-2 (*rademacher*), 31

theorem-3

theorem-3 (*rademacher*), 32

total-var

total-var (*rademacher*), 31

unique-prod

unique-prod (*fubini*), 65

upper-density

upper-density (*hausdorff_measure*), 37

vector-measure

vector-measure (*standard_theorems*), 60

vitali-cover

vitali-cover (*covering*), 25

vitali-set

vitali-set (*measures*), 10

vitali-set-properties

vitali-set-properties (*hausdorff_measure*), 39

vitali-theorem

vitali-theorem (*covering*), 26

weak*-compact

weak*-compact (*daniell*), 22

weak-convergence

weak-convergence (*daniell*), 22

Symbols

5r covering lemma, 25

A

Almost everywhere, 8

Approximate tangent plane, 41

B

Besicovitch-Federer projection theorem, 48

C

Cantor set, 10

Carathéodory construction, 7

Carathéodory criterion, 6

D

Dominated convergence theorem, 14

E

Egorov's theorem, 57

F

Fubini's theorem, 64

Function

 Absolutely continuous, 31

 Borel, 13

 Simple, 13

H

Hardy-Littlewood maximal function, 27

Hardy-Littlewood maximal inequality, 27

Hausdorff density, 37

Hausdorff measure, 10, 37

I

Integral, 13

L

Lattice of functions, 17

Lebesgue decomposition theorem, 59

Lebesgue density theorem, 29

Lebesgue differentiation theorem, 28

Lusin's theorem, 57

M

Measurable function, 13

Measurable set, 3

Measure, 3

 Absolutely continuous, 58

 Borel regular, 7

 Counting, 9

 Dirac, 9

 Doubling, 26

 Lebesgue, 8

 Push forward, 8

 Radon, 7

 Sigma finite, 7

 Singular, 58

 Total variation, 31

Monotone convergence theorem, 14

P

Purely unrectifiable set, 47

R

Rademacher's theorem, 32

Radon-Nikodym theorem, 58

Rectifiable set, 41

Restriction measure, 10

Reverse Fatou, 14

Riesz representation theorem, 22

S

Sard's theorem, 41

Sigma algebra, 5

V

Vitali cover, 25

Vitali covering theorem, 26

Vitali set, 10

W

Weak* convergence, 22