METRIC GEOMETRY, FALL 2018
EXERCISE 2, IDEAS FOR SOLUTIONS.

Exercise 1. *i)* Suppose $x \in C_\varepsilon$. By definition, there exists $\delta > 0$ so that, for all $y, z \in U(x, \delta)$, we have $|f(y) - f(z)| < \varepsilon$. If $x' \in U(x, \delta)$, we may pick $\delta' = \delta - d(x, x')$, and now since $U(x', \delta') \subset U(x, \delta)$, it is easily seen that $x' \in C_\varepsilon$. We conclude that $U(x, \delta) \subset C_\varepsilon$, resulting in openness of C_ε since x was arbitrary.

ii) We claim that the points of continuity for f are exactly $\bigcap_{\varepsilon>0} C_\varepsilon$. The claim then follows, since it is easily seen that $\bigcap_{\varepsilon>0} C_\varepsilon = \bigcap_{i=1}^{\infty} C_{1/i}$.

Recall that by definition, f is continuous at x if, for every $\varepsilon > 0$, there is $\delta > 0$ so that for all $y \in U(x, \delta)$, we have $|f(y) - f(x)| < \varepsilon$. Hence, by selecting $z = x$ in the definition of C_ε, we have that $x \in C_\varepsilon$ implies the continuity condition for ε. Moreover, by using the triangle inequality, we have that the continuity condition for ε implies $x \in C_{2\varepsilon}$. It follows that f is continuous at exactly $\bigcap_{\varepsilon>0} C_\varepsilon$.

iii) We then consider a $f : \mathbb{R} \to \mathbb{R}$. Let C be the set of points of continuity of f, which by ii) we know to be Borel. For $x \in \mathbb{R}$, let F_x be the function

$$F_x(t) = \frac{f(x+t) - f(x)}{t}.$$

That is, $f'(x)$ exists if and only if $\lim_{t\to 0} F_x(t)$ exists.

We define the set $D_{\varepsilon,\delta}$ for $\varepsilon, \delta > 0$ to be the set of points $x \in \mathbb{R}$ for which for all $t, s \in B(0, \delta) \setminus \{0\}$ we have

$$|F_x(t) - F_x(s)| \leq \varepsilon.$$

It follows that $D_{\varepsilon,\delta} \cap C$ is closed in the relative topology of C: for fixed $t, s \in B(0, \delta)$ we have that $x \mapsto |F_x(t) - F_x(s)|$ is continuous on C, and $D_{\varepsilon,\delta} \cap C$ is therefore an intersection of preimages of $B(0, \varepsilon)$ under continuous maps. Hence, we have $D_{\varepsilon,\delta} \cap C = A \cap C$ for some closed A, and therefore $D_{\varepsilon,\delta} \cap C$ is Borel.

We now define

$$D = \left(\bigcap_{\varepsilon>0} \bigcup_{\delta>0} (D_{\varepsilon,\delta} \cap C) \right).$$

That is, $x \in D$ if and only if $x \in C$ and for every $\varepsilon > 0$, there is a $\delta > 0$ for which $|F_x(t) - F_x(s)| \leq \varepsilon$ for all $t, s \in B(0, \delta)$. Note that since $D_{\varepsilon,\delta}$ increases as δ tends to zero and decreases as ε tends to zero, we obtain the same D if we replace the union and intersection in the definition of D by countable ones, as was done in ii). Hence, by our previous observations, D is a Borel set.

If f is differentiable at x, then $x \in C$, and it is therefore an easy application of triangle inequality to show that $x \in D$. Conversely, if $x \in D$, it is relatively straightforward to see using the completeness of $\mathbb{R}$ that $F_x(t)$ has a limit as $t \to 0$. Hence, D is precisely the set of points of differentiability for f.

Alternate approach for iii) It is also possible to instead let $D_{\varepsilon,\delta,q}$, for $\varepsilon, \delta > 0$ and $q \in \mathbb{Q}$, be the set of points x where for all $t \in B(0, \delta) \setminus \{0\}$ we have

$$|F_x(t) - q| \leq \varepsilon.$$

Then, we may define

$$D = \left(\bigcap_{\varepsilon>0} \bigcup_{\delta>0} \bigcup_{q\in\mathbb{Q}} (D_{\varepsilon,\delta,q} \cap C) \right).$$

We note again that replacing ε and δ with sequences tending to 0 yields the same D, and hence D is Borel as previously.

If f is differentiable at x, then by approximating $f'(x)$ with rational q and using the triangle inequality, it follows that $x \in D$. Conversely, if $x \in D$, we find q_i so that $|F_x(t) - q_i| \leq 1/i$ for each $t \in B(x, \delta_i) \setminus \{0\}$. It follows by triangle inequality that $q_j \in B(q_i, 2/i)$ for $j \geq i$, and therefore the sequence q_i is Cauchy. Hence, by completeness of $\mathbb{R}$ we now have a limit $q_i \to y$, and it is straightforward to show that $F_x(t) \to y$ as $t \to 0$, proving again that D is the set of differentiability points of f.

Exercise 2. *Remark:* There's a small mistake in the problem statement in that the set S should be assumed to be measurable. Suppose this assumption is not in place. Then the porous set S may be non-Borel, as subsets of porous sets are clearly porous, and the Cantor 1/3-set is a porous set containing more subsets than there are Borel sets.

Whether the porous set S is always contained in a zero measure Borel set (and hence measurable and of zero measure if we assume μ to be the completion) seems to be a more complicated question. For some definitions of porosity, it can be shown that the closure of a porous set is porous, and we could then apply the solution of the measurable version of the exercise to the closure of S (see eg. the definition of a porous set in Wikipedia). However, for the definition given in the exercise, the closure of a porous set can be non-porous (a counterexample can be found by a suitable increasing sequence in $[0, 1]$ which tends to 1 and increases in density at a fast enough rate). As of now, I have no definitive answer for whether the given porous set S is always contained in a zero-measure Borel set even if we don't assume measurability.

Solution assuming that S is measurable: We suppose to the contrary that X is a doubling m.m.s. and $S \subset X$ is porous with $\mu(S) > 0$. By the Lebesgue density theorem, we can select a density point $x \in S$ for which $\lim_{r\to 0} \mu(B(x, r) \cap S)/\mu(B(x, r)) = 1$. If $\varepsilon > 0$, then for all $0 < r < r_\varepsilon$ we have $\mu(B(x, r) \cap S) > (1 - \varepsilon)\mu(B(x, r))$.

Now, by definition of porousness, we find $x_n \in B(x, r_\varepsilon/2)$ for which $B(x_n, \lambda d(x, x_n)) \cap S = \emptyset$. Clearly $\lambda < 1$, so if we pick $r = (1 + \lambda)d(x_n, x)$, we have $r < r_\varepsilon$. Now

$$\mu(B(x, r) \cap S) \leq \mu(B(x, r)) - \mu\left(B\left(x_n, \frac{\lambda}{1+\lambda} r\right)\right).$$

We may pick $i_\lambda > 1$ so that $2^{-i_\lambda} < \lambda/(1+\lambda)$. Now by doubling, we get

$$\begin{aligned}\mu(B(x,r)\cap S) &\leq \mu(B(x,r)) - \mu\left(B\left(x_n, \frac{\lambda}{1+\lambda}r\right)\right)\\ &\leq \mu(B(x,r)) - \mu(B(x_n, 2^{-i_\lambda}r))\\ &\leq \mu(B(x,r)) - C^{-i_\lambda}\mu(B(x_n,r)).\end{aligned}$$

Moreover, since $r > d(x_n,x)$, we have $B(x,r) \subset B(x_n,2r)$, and we may continue

$$\begin{aligned}\mu(B(x,r)\cap S) &\leq \mu(B(x,r)) - C^{-(1+i_\lambda)}\mu(B(x_n,2r))\\ &\leq \mu(B(x,r)) - C^{-(1+i_\lambda)}\mu(B(x,r)).\end{aligned}$$

Hence, this is a contradiction if we pick $\varepsilon < C^{-(1+i_\lambda)}$. Note that with the fixing of variables written in this order, we have to be very careful that $C^{-(1+i_\lambda)}$ does not depend on ε: it depends on the doubling constant $C = C(X,\mu)$ and the $\lambda = \lambda(X,\mu,x)$ from the definition of porousness.

Exercise 3. We recall the notation

$$\mathcal{H}^\alpha(S) = \lim_{\delta\to 0}\mathcal{H}^\alpha_\delta(S),$$

where

$$\mathcal{H}^\alpha_\delta(S) = \inf\left\{\sum_{i=1}^\infty d(U_i)^\alpha : S \subset \bigcup_{i=1}^\infty U_i, d(U_i) < \delta\right\}.$$

i) Suppose that $f : (X,d) \to (Y,d')$ is L-lipschitz. Pick a cover $\{U_i\}$ for S with $d(U_i) < \delta$ and $\sum_{i=1}^\infty d(U_i)^\alpha \leq \mathcal{H}^\alpha_\delta(S) + \varepsilon$. Now, $\{f(U_i)\}$ is a cover of $f(S)$, $d'(f(U_i)) < L\delta$, and $\sum_{i=1}^\infty d'(f(U_i))^\alpha \leq L^\alpha(\mathcal{H}^\alpha_\delta(S)+\varepsilon)$. By letting $\varepsilon \to 0$, we obtain $\mathcal{H}^\alpha_{L\delta}(f(S)) \leq L^\alpha\mathcal{H}^\alpha_\delta(S)$, which gives the desired result in the limit.

ii) Suppose next that $R > 0$ and for each $x \in S$ and every $y \in U(x,R)$, we have $d'(f(x),f(y)) < Ld(x,y)$. Suppose $A \subset S$ and $d(A) < R$. Then $A \subset U(x,R)$ for every $x \in A$, and we have

$$d'(f(U_i)) = \sup_{x\in A}\sup_{y\in A} d'(f(x),f(y)) \leq L\sup_{x\in A}\sup_{y\in A} d(x,y) = Ld(U_i).$$

Hence, if $\delta < R$, we may repeat the estimates done in i) but with the sets U_i replaced by $U_i \cap S$. We obtain the same result in the limit.

iii) Suppose then that $\mathrm{Lip}(f,x)$ is bounded from above by L. From this, it follows that for every $x \in X$ and every $\varepsilon > 0$ there exists a $R_{x,\varepsilon} > 0$ so that if $y \in B(x,R_{x,\varepsilon})$, we have $d'(f(x),f(y)) < (L+\varepsilon)d(x,y)$.

It was given in the exercise that we may assume $x \mapsto \mathrm{Lip}(f,x)$ is Borel, along with similar statements. Here, we demonstrate how to prove the claim under the following assumption: we may select $R_{x,\varepsilon}$ so that, for every ε, the function $x \mapsto R_{x,\varepsilon}$ is Borel.

For $R,\varepsilon > 0$, let $X_{\varepsilon,R} = \{x \in X : R_{x,\varepsilon} \geq R\}$. We have $X = \bigcup_{k=1}^\infty X_{\varepsilon,1/k}$. The sequence $X_{\varepsilon,1/k}$ is also increasing w.r.t. k. We denote for convenience that $X_{\varepsilon,1/0} = \emptyset$. Now we may use part ii) on these sets to obtain

$$\begin{aligned}\mathcal{H}^\alpha(f(X)) &\le \sum_{k=1}^{\infty} \mathcal{H}^\alpha(f(X_{\varepsilon,1/k} \setminus X_{\varepsilon,1/(k-1)}))\\ &\le \sum_{k=1}^{\infty} (L+\varepsilon)^\alpha \mathcal{H}^\alpha(X_{\varepsilon,1/k} \setminus X_{\varepsilon,1/(k-1)})\\ &= (L+\varepsilon)^\alpha \mathcal{H}^\alpha(X).\end{aligned}$$

Here, the last step requires our assumption that $x \mapsto R_{x,\varepsilon}$ is Borel, as that implies that the sets $X_{\varepsilon,R}$ are Borel and additivity therefore applies. By doing this for all $\varepsilon > 0$, the claim follows.

Exercise 4. *i)* Recall that for $\gamma\colon [a,b] \to X$, $\operatorname{len}(\gamma)$ is defined by

$$\operatorname{len}(\gamma) = \sup\left\{ \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) : a = t_0 < t_1 < \ldots < t_n = b \right\}.$$

By selecting $t_0 = a, t_1 = b$, we immediately obtain $\operatorname{len}(\gamma) \ge d(\gamma(a), \gamma(b))$.

ii) Before starting, we point out two facts which are of use. First, if $x, y, z \in [a,b]$ with $x < y < z$, then

$$\operatorname{len}(\gamma|_{[x,z]}) = \operatorname{len}(\gamma|_{[x,y]}) + \operatorname{len}(\gamma|_{[y,z]}).$$

This is not too difficult to show using the definition of len. The other fact is that if L is the Lipschitz constant of γ, we have

$$\operatorname{len}(\gamma|_{[x,y]}) \le L(y-x).$$

This is a straightforward Lipschitz estimate on the definition of len.

Recall that $l_\gamma\colon [a,b] \to [0, \operatorname{len}(\gamma)]$ is defined by $l_\gamma(t) = \operatorname{len}(\gamma|_{[a,t]})$, and that the arc-length parametrization is a 1-Lipschitz $\gamma^*\colon [0, \operatorname{len}(\gamma)] \to X$ with $\gamma^* \circ l_\gamma = \gamma$.

It is easily seen that l_γ is increasing, $l_\gamma(a) = 0$ and $l_\gamma(b) = \operatorname{len}(\gamma)$. Furthermore, if $a \le t \le t' \le b$, our two previously mentioned useful facts yield

$$l_\gamma(t') - l_\gamma(t) = \operatorname{len}(\gamma|_{[t,t']}) \le L(t'-t),$$

where L is the Lipschitz constant of γ. Hence, l_γ is continuous, and therefore also surjective.

We define a map $\sigma_\gamma\colon [0, \operatorname{len}(\gamma)] \to [a,b]$ by

$$\sigma_\gamma(c) = \inf\left\{ s \in l_\gamma^{-1}\{c\} \right\},$$

and set $\gamma^* = \gamma \circ \sigma_\gamma$. We want to show that $\gamma^* \circ l_\gamma = \gamma$ and that γ^* is 1-Lipschitz.

Suppose that $t \in [a,b]$. Note that since l_γ is continuous, $l_\gamma^{-1}\{l_\gamma(t)\}$ is closed. Hence, $\sigma_\gamma(l_\gamma(t))$ is the least $s \in [a,b]$ for which $l_\gamma(s) = l_\gamma(t)$. Hence, we have by the first useful fact from the beginning of this part that $\operatorname{len}(\gamma|_{[s,t]}) = 0$, and therefore by part i), γ is constant on $[s,t]$. Hence,

$$\gamma^*(l_\gamma(t)) = \gamma(\sigma_\gamma(l_\gamma(t))) = \gamma(s) = \gamma(t),$$

and twe obtain the desired $\gamma^* \circ l_\gamma = \gamma$.

It remains to show that γ^* is 1-Lipschitz. For this, we note that by i) and our first useful fact,

$$\begin{aligned} d(\gamma^*(l_\gamma(t)), \gamma^*(l_\gamma(t'))) &= d(\gamma(t), \gamma(t')) \\ &\leq \operatorname{len}(\gamma|_{[t,t']}) \\ &= \left| l_\gamma(t) - l_\gamma(t') \right|. \end{aligned}$$

Since l_γ is surjective, this is the 1-Lipschitz condition for γ^*.

iii) We note that since σ_γ and l_γ are non-decreasing, we easily obtain using the definition of len that

$$\begin{aligned} \operatorname{len}(\gamma^*|_{[s,t]}) &= \operatorname{len}((\gamma \circ \sigma_\gamma)|_{[s,t]}) \\ &\leq \operatorname{len}(\gamma|_{[\sigma_\gamma(s), \sigma_\gamma(t)]}) \\ &= \operatorname{len}((\gamma^* \circ l_\gamma)|_{[\sigma_\gamma(s), \sigma_\gamma(t)]}) \\ &\leq \operatorname{len}(\gamma^*|_{[l_\gamma(\sigma_\gamma(s)), l_\gamma(\sigma_\gamma(t))]}) \\ &= \operatorname{len}(\gamma^*|_{[s,t]}). \end{aligned}$$

Hence, by definition of l_γ and the first useful fact, we obtain

$$\begin{aligned} \operatorname{len}(\gamma^*|_{[s,t]}) &= \operatorname{len}(\gamma|_{[\sigma_\gamma(s), \sigma_\gamma(t)]}) \\ &= l_\gamma(\sigma_\gamma(t)) - l_\gamma(\sigma_\gamma(s)) \\ &= t - s, \end{aligned}$$

which proves the claim.

iv) Note that, since γ is injective and l_γ is surjective, we may by $\gamma = \gamma^* \circ l_\gamma$ deduce that γ^* is also injective. We begin by estabilishing an auxiliary estimate that given $0 \leq s \leq t \leq \operatorname{len}(\gamma)$, we have

$$\mathcal{H}^1(\gamma^*([s,t])) \geq d(\gamma^*(t), \gamma^*(s)).$$

Suppose that this is not true. Then by definition of $\mathcal{H}^1$, there is a countable collection $\{S_i\}$ covering $\gamma^*([s,t])$ so that $\sum_i d(S_i) < d(\gamma^*(t), \gamma^*(s))$. We may assume the S_i are open by replacing them with suitable open balls around S_i, with radii chosen via the standard $2^{-i}\varepsilon$-trick so that the sum of diameters stays below $d(\gamma^*(t), \gamma^*(s))$. Then we may assume there are only finitely many U_i by compactness of $\gamma^*([s,t])$.

Suppose $\gamma^*(s) \in U_{i_1}$. If $\gamma^*(t)$ is not in U_{i_1}, we may by connectedness of $\gamma^*([s,t])$ select U_{i_2} which intersects U_{i_1}. If this is still not the case, we may again select U_{i_3} which intersects $U_{i_1} \cup U_{i_2}$, and continue this until $\gamma^*(t) \in U_{i_k}$ (which happens since we only have finitely many sets to consider). Note that if $U_i \cap U_j \neq \emptyset$, we have $d(U_i \cup U_j) \leq d(U_i) + d(U_j)$. By this, we get by an inductive calculation that $d(U_{i_1} \cup \ldots U_{i_k}) \leq d(U_{i_1}) + \ldots + d(U_{i_k}) < d(\gamma^*(s), \gamma^*(t))$. this is a contradiction.

With that obvious-but-surprisingly-clumsy-to-write lemma covered, we actually prove the claim. By iii), exercise 3, and the fact that γ^* is 1-Lipschitz, we have

$$\mathcal{H}^1(\gamma^*([s,t])) \leq \mathcal{H}^1([s,t]) = t - s = \operatorname{len}(\gamma^*|_{[s,t]}).$$

Suppose then that $s = x_0 < x_1 < \ldots < x_m = t$. Our lemma gives

$$\sum_{i=1}^m d(\gamma^*(x_i), \gamma^*(x_{i-1})) \leq \sum_{i=1}^m \mathcal{H}^1(\gamma^*([x_{i-1}, x_i]))$$

Since γ^* is injective, $\gamma^*([x_{i-1}, x_i])$ overlap only in $\{x_1, \dots, x_{m-1}\}$, which has $\mathcal{H}^1$-measure zero. Hence, we may use additivity to conclude that

$$\sum_{i=1}^{m} d(\gamma^*(x_i), \gamma^*(x_{i-1})) \leq \mathcal{H}^1(\gamma^*([s,t])).$$

Now, $\text{len}(\gamma^*|_{[s,t]}) \leq \mathcal{H}^1(\gamma^*([s,t]))$ follows by taking the supremum over all such partitions, and the claim then follows.

v) By iii) and iv), we have $\mathcal{H}^1(\gamma^*(I)) = \mathcal{L}^1(I)$ for every closed interval $I \subset [0, \text{len}(\gamma)]$. Let Σ be the collection of sets $S \subset [0, \text{len}(\gamma)]$ for which $\mathcal{H}^1(\gamma^*(S)) = \mathcal{L}^1(S)$. It is easy to see that Σ is a σ-algebra containing all closed intervals in $[0, \text{len}(\gamma)]$, and therefore it must contain all Borel sets of $[0, \text{len}(\gamma)]$.

vi) A simple example is given by a path which loops around S^1 twice: by using parts iii) and iv) to its injective subpaths, we easily see that $\mathcal{H}^1$ yields 2π, but len yields 4π.

Exercise 5. Suppose g is any continuous extension of f to $\overline{S}$. If $x \in \overline{S}$, then there's a sequence $(x_i) \in S$ for which $x_i \to x$. Now, by continuity, $g(x) = \lim_{i\to\infty} f(x_i)$, which shows the uniqueness of such extensions of f.

It remains to show that the above $f(x) = \lim_{i\to\infty} f(x_i)$ gives a well-defined L-lipschitz extension of f to $\overline{S}$. By continuity of f, it is clear that this is well defined and agrees with the original f on S.

Suppose then that $x \in \partial S$ and $(x_i) \in S$ with $x_i \to x$. Since f is L-lipschitz, we have $d'(f(x_i), f(x_j)) \leq L d(x_i, x_j)$. Therefore, since (x_j) is Cauchy, the sequence $(f(x_j))$ is also Cauchy. By completeness of Y, the limit $\lim_{i\to\infty} f(x_i)$ therefore exists. Moreover, if $(x_i') \in S$ is another sequence with $x_i' \to x$, then $d'(f(x_i), f(x_i')) \leq L d(x_i, x_i')$ yields that $\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} f(x_i')$. Hence, the limit is independent of the sequence selected, and we obtain a well-defined extension of f to a function $\overline{S} \to Y$.

Remains to show that the extended f is L-Lipschitz. For this, let $x, y \in \overline{S}$, and $(x_i), (y_i) \in S$ with $x_i \to x$ and $y_i \to y$. Now, by continuity of d' and d, we obtain $d'(f(x), f(y)) = \lim_{i\to\infty} d'(f(x_i), f(y_i)) \leq \lim_{i\to\infty} L d(x_i, y_i) = L d(x, y)$, showing that the extension is L-Lipschitz.

Finally, consider the case where Y is not complete. We may select $X = [0,1]$, $S = Y = (0,1]$, and $f = \text{id}_{(0,1]}$. It is then clear that f is 1-Lipschitz but has no continuous extension to X, and therefore we obtain a counterexample.

Exercise 6. Suppose $x_1, x_2 \in X$ and $\varepsilon > 0$. Then we find n_1, n_2 so that $d'(f_n(x_1), f(x_1)) < \varepsilon$ whenever $n \geq n_1$ and $d'(f_n(x_2), f(x_2)) < \varepsilon$ whenever $n \geq n_2$. Pick $n = \max(n_1, n_2)$, and use triangle inequality along with the fact that f_n is L-Lipschitz to obtain $d'(f(x_1), f(x_2)) \leq L d(x_1, x_2) + 2\varepsilon$. By letting $\varepsilon \to 0$, the claim follows.

Exercise 7. The only question regarding the well-definedness of ι is whether $\iota(x)$ is in ℓ^∞. This follows from the fact that $|\iota(x)_i| = |d(x, x_i) - d(x_0, x_i)| \leq d(x, x_0)$ by the left side of the triangle inequality.

Suppose then that $x, y \in X$. We have $\iota(x) - \iota(y) = (d(x, x_i) - d(y, x_i))$, so by the same estimate as previously, $\|\iota(x) - \iota(y)\|_\infty \leq d(x, y)$. Moreover, since (x_i) is dense, we may select a subsequence x_{i_j} converging to y, and by continuity of d, we get $\lim_{j\to\infty} d(x, x_{i_j}) - d(y, x_{i_j}) = d(x, y) - d(y, y) = d(x, y)$. Hence, $\|\iota(x) - \iota(y)\|_\infty = d(x, y)$, and therefore ι is an isometry.

Now, let $\tilde{X}$ be the closed linear span of $\iota(X)$. By definition, it is clear that $\tilde{X}$ is closed, and by continuity of linear operations it is also easy to see that $\tilde{X}$ is a linear subspace of ℓ^∞. We wish to show that $\tilde{X}$ has a dense countable set F. We select

$$F = \{q_1\iota(x_{i_1}) + \ldots + q_m\iota(x_{i_m}) : q_j \in \mathbb{Q}\}.$$

It is rather simple to see that our F is countable. Moreover, if y is in the linear span of $\iota(X)$, it is of the form $a_1\iota(y_1) + \ldots + a_m\iota(y_m)$ for some $a_j \in \mathbb{R}$ and $y_j \in X$. If we approximate y_j by elements of (x_i), the fact that ι is isometric yields the same approximation on the image side. Hence, it follows easily that F is dense in the linear span of $\iota(X)$, and therefore also its closure, $\tilde{X}$.