METRIC GEOMETRY, FALL 2018
EXERCISE 3, IDEAS FOR SOLUTIONS.

Exercise 1. Recall that $\rho\colon X \to \mathbb{R}$ is an upper gradient of f if for all $x, y \in X$ and every rectifiable curve γ from x to y we have

$$|f(x) - f(y)| \leq \int_\gamma \rho \,\mathrm{d}s,$$

where the latter integral is defined via the arc-length parametrization

$$\int_\gamma \rho \,\mathrm{d}s = \int_0^{\mathrm{len}(\gamma)} \rho \circ \gamma^*(t) \,\mathrm{d}t.$$

Recall also that

$$\mathrm{Lip}(f, x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We will denote by Lip_f the function $x \mapsto \mathrm{Lip}(f, x)$. As in the previous exercises, we may assume this function is Borel.

Method 1: Suppose γ starts from x and ends at y. Since f is Lipschitz, we have that $f \circ \gamma^*$ is a Lipschitz real function. Therefore, it is absolutely continuous, and the fundamental theorem of calculus holds for it:

$$f(y) - f(x) = \int_0^{\mathrm{len}(\gamma)} (f \circ \gamma^*)'(t) \,\mathrm{d}t.$$

By the triangle inequality for integrals ($\left|\int_A g\right| \leq \int_A |g|$), this of course implies

$$|f(y) - f(x)| \leq \int_0^{\mathrm{len}(\gamma)} \left|(f \circ \gamma^*)'(t)\right| \mathrm{d}t.$$

Hence, it is sufficient to prove that $|(f \circ \gamma^*)'(t)| \leq (\mathrm{Lip}_f \circ \gamma^*)(t)$ almost everywhere.

Suppose then that the derivative exists at t (which holds almost everywhere), let $\varepsilon > 0$, and let $t_i \to t$. Since γ^* is continuous, $\gamma^*(t_i)$ gets close to $\gamma^*(t)$ as i increases, and by definition of Lip_f we therefore have for large enough i that

$$\frac{|(f \circ \gamma^*)(t_i) - (f \circ \gamma^*)(t)|}{|t_i - t|} \leq \frac{(\mathrm{Lip}_f(\gamma^*(t)) + \varepsilon)\, |\gamma^*(t_i) - \gamma^*(t)|}{|t_i - t|}$$

Moreover, since γ^* is 1-Lipschitz, we have

$$\frac{|\gamma^*(t_i) - \gamma^*(t)|}{|t_i - t|} \leq 1.$$

Hence, by taking the limit we obtain that $|(f \circ \gamma^*)'(t)| \leq (\mathrm{Lip}_f \circ \gamma^*)(t) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain the result.

Method 2: First, suppose that g is a simple Borel function with $0 \leq g \leq \mathrm{Lip}_f$. Since γ^* is 1-Lipschitz, by Ex. 3 i) of the last set of exercises, we easily obtain

$$\int_{\gamma^*[0,\mathrm{len}(\gamma)]} g \, \mathrm{d}\mathcal{H}^1 \leq \int_0^{\mathrm{len}(\gamma)} g \circ \gamma^*(t) \, \mathrm{d}t.$$

Since Lip_f is assumed to be Borel, with a suitable sequence of simple Borel functions and monotone convergence it follows that

$$\int_{\gamma^*[0,\mathrm{len}(\gamma)]} \mathrm{Lip}_f \, \mathrm{d}\mathcal{H}^1 \leq \int_0^{\mathrm{len}(\gamma)} \mathrm{Lip}_f \circ \gamma^*(t) \, \mathrm{d}t.$$

Next, note that since f is L-Lipschitz for some L, we have $\mathrm{Lip}_f \leq L$. Suppose that h is a simple Borel function on $\gamma^*[0, \mathrm{len}(\gamma)]$ with $\mathrm{Lip}_f \leq h \leq L$. We may write h in the form $h = \sum_i L_i \chi_{S_i}$, where $S_i \subset \gamma^*[0, \mathrm{len}(\gamma)]$ are a disjoint partition of $\gamma^*[0, \mathrm{len}(\gamma)]$ and $\mathrm{Lip}_f |_{S_i} \leq L_i$. By Ex. 3 iii) of the previous exercises, we may now deduce $\mathcal{H}^1(f(S_i)) \leq L_i \mathcal{H}^1(S_i)$, from which it follows that

$$\mathcal{H}^1(f \circ \gamma^*[0, \mathrm{len}(\gamma)]) \leq \sum_i \mathcal{H}^1(f(S_i)) \leq \int_{\gamma^*[0,\mathrm{len}(\gamma)]} h \, \mathrm{d}\mathcal{H}^1.$$

Now, we may use uniformly bounded convergence to a suitable descending sequence of simple Borel functions h_i converging pointwise to Lip_f, and as a result we obtain

$$\mathcal{H}^1(f \circ \gamma^*[0, \mathrm{len}(\gamma)]) \leq \int_{\gamma^*[0,\mathrm{len}(\gamma)]} \mathrm{Lip}_f \, \mathrm{d}\mathcal{H}^1.$$

Finally, we note that $f \circ \gamma^*[0, \mathrm{len}(\gamma)]$ is an interval containing the points $f(x)$ and $f(y)$. Hence, we have

$$|f(x) - f(y)| \leq \mathcal{H}^1(f \circ \gamma^*[0, \mathrm{len}(\gamma)]),$$

and chaining the estimates obtained so far proves the claim.

Exercise 2. We have

$$D_\varepsilon = ([-3, -1] \times [-1, 1]) \cup ([-2, 2] \times [-\varepsilon, \varepsilon]) \cup ([1, 3] \times [-1, 1]).$$

As suggested in the exercise, we consider the function

$$f(x, y) = \begin{cases} -1, & x \leq -1 \\ x, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

on D_ε. Then we clearly have the upper gradient

$$\rho(x, y) = \begin{cases} 0, & x \leq -1 \text{ or } x \geq 1 \\ 1, & -1 < x < 1 \end{cases}$$

for f (one may for example use the conclusion of the previous exercise for this).

We now compute

$$\int_{D_\varepsilon} \rho \, \mathrm{d}\mathcal{L}^2 = \mathcal{L}^2(]-1, 1[\times [-\varepsilon, \varepsilon]) = 4\varepsilon.$$

Furthermore, it is easily seen that $f_{D_\varepsilon} = 0$, and therefore

$$\begin{aligned}\int_{D_\varepsilon} |f - f_B| \,\mathrm{d}\mathcal{L}^2 &> \mathcal{L}^2(([-3,-1]\times[-1,1]) \cup ([1,3]\times[-1,1])) \\ &= 8\end{aligned}$$

Moreover, it is easily seen that $D_\varepsilon = B_{D_\varepsilon}((0,0),4)$. Therefore, the 1-PI constant of D_ε is greater than $8/(4\cdot 4\varepsilon) = 1/(2\varepsilon)$. As ε shrinks, this becomes arbitrarily large.

Exercise 3. *Note: As mentioned in the exercise sessions, it turns out that the PI-space part of this exercise is "harder than expected", and as a result, it has been pushed to a later set of exercises where more detailed instructions will be given. We nevertheless give the solution for the doubling part here.*

We suppose that (X, d, μ) is a m.m.s. and that $\iota : X \to Y$ is a L-bilipschitz map, where (Y, d') is a metric space. We equip Y with the push-forward measure $\nu = \iota_*\mu$: recall that the push-forward of measures is defined by $\iota_*\mu(S) = \mu(\iota^{-1}S)$ (in our case for Borel S).

Suppose now that μ is doubling with constant λ. Then

$$\begin{aligned}\nu(B(y,2r)) &= \mu(\iota^{-1}B(y,2r)) \\ &\leq \mu(B(\iota^{-1}(y), 2Lr)) \\ &\leq \lambda^{\lceil \log_2(L^2)\rceil+1}\mu(B(\iota^{-1}(y), r/L)) \\ &= \lambda^{\lceil \log_2(L^2)\rceil+1}\nu(\iota(B(\iota^{-1}(y), r/L))) \\ &\leq \lambda^{\lceil \log_2(L^2)\rceil+1}\nu(B(y,r)).\end{aligned}$$

Hence, ν is doubling, with constant $\lambda' = \lambda^{\lceil \log_2(L^2)\rceil+1}$. (Note that here, $\lceil x\rceil$ is the ceiling function, which maps x to the least integer $n \geq x$).

Exercise 4. *i)* We have equipped $\mathbb{R}$ with a chart $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ which maps x to (x, x). Let $x_0 \in \mathbb{R}^n$ and suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. If $t \in \mathbb{R}$ is some real number, we define $D_t f(x_0) = (tf'(x_0), (1-t)f'(x_0)) \in \mathbb{R}^2$. Now

$$\begin{aligned}&\limsup_{x\to x_0} \frac{|f(x) - f(x_0) - D_t f(x_0)\cdot(\varphi(x) - \varphi(x_0))|}{|x - x_0|} \\ &= \limsup_{x\to x_0} \frac{|f(x) - f(x_0) - (t + 1 - t)f'(x_0)(x - x_0)|}{|x - x_0|} \\ &= 0.\end{aligned}$$

Hence, a Lipschitz $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable with respect to φ (with infinitely many derivatives) at every point where it's differentiable in the usual sense. Hence, $\mathbb{R}$ equipped with φ is a Lipschitz differentiability space (under the definition which disregards the existence of infinitely many derivatives).

ii) We suppose now that $f\colon X \to \mathbb{R}$ is Lipschitz, and f has two derivatives $D_1 \neq D_2$ at x_0 with respect to the chart $\varphi\colon X \to \mathbb{R}^n$. We then have by a simple triangle inequality application that

$$\begin{aligned}
&\limsup_{x\to x_0} \frac{|(D_1 - D_2)\cdot(\varphi(x)-\varphi(x_0))|}{d(x,x_0)} \\
&\qquad \leq \limsup_{x\to x_0} \frac{|f(x)-f(x_0)-D_1\cdot(\varphi(x)-\varphi(x_0))|}{d(x,x_0)} \\
&\qquad\quad + \limsup_{x\to x_0} \frac{|f(x)-f(x_0)-D_2\cdot(\varphi(x)-\varphi(x_0))|}{d(x,x_0)} \\
&\qquad = 0.
\end{aligned}$$

iii) Since $D_1 - D_2 \neq 0$, we may pick some component $i \in \{1,\dots,n\}$ so that $(D_1 - D_2)_i \neq 0$. Similarly as with the charts, we denote by $(D_1 - D_2)^i$ the element of $\mathbb{R}^{n-1}$ with the same components as $D_1 - D_2$ except with the i:th one skipped. Then

$$\begin{aligned}
&\limsup_{x\to x_0} \frac{\left|\varphi_i(x)-\varphi_i(x_0) - \frac{-(D_1-D_2)^i}{(D_1-D_2)_i}\cdot(\varphi^i(x)-\varphi^i(x_0))\right|}{d(x,x_0)} \\
&\quad = \frac{1}{|(D_1-D_2)_i|}\limsup_{x\to x_0} \frac{|(D_1 - D_2)\cdot(\varphi(x)-\varphi(x_0))|}{d(x,x_0)} \\
&\quad = 0.
\end{aligned}$$

Hence, φ_i is differentiable at x with respect to φ^i, where the derivative at x_0 is $D = -(D_1 - D_2)^i/(D_1 - D_2)_i$.

iv) Using now the D we derived in the previous part, we now estimate that

$$\begin{aligned}
&\limsup_{x\to x_0} \frac{\left|f(x)-f(x_0)-((D_1)^i+(D_1)_iD)\cdot(\varphi^i(x)-\varphi^i(x_0))\right|}{d(x,x_0)} \\
&\quad \leq \limsup_{x\to x_0} \frac{|f(x)-f(x_0)-D_1\cdot(\varphi(x)-\varphi(x_0))|}{d(x,x_0)} \\
&\quad\quad + \limsup_{x\to x_0} \frac{\left|D_1\cdot(\varphi(x)-\varphi(x_0))-((D_1)^i+(D_1)_iD)\cdot(\varphi^i(x)-\varphi^i(x_0))\right|}{d(x,x_0)}.
\end{aligned}$$

The first term goes to zero as D_1 is a derivative of f at x_0 w.r.t. φ. For the second term, note that

$$\begin{aligned}
&D_1\cdot(\varphi(x)-\varphi(x_0))-((D_1)^i+(D_1)_iD)\cdot(\varphi^i(x)-\varphi^i(x_0)) \\
&\quad = (D_1)_i(\varphi_i(x)-\varphi_i(x_0)) + (D_1)^i\cdot(\varphi^i(x)-\varphi^i(x_0)) \\
&\qquad - (D_1)^i\cdot(\varphi^i(x)-\varphi^i(x_0)) - (D_1)_iD\cdot(\varphi^i(x)-\varphi^i(x_0)) \\
&\quad = (D_1)_i(\varphi_i(x)-\varphi_i(x_0)) - (D_1)_iD\cdot(\varphi^i(x)-\varphi^i(x_0)) \\
&\quad = (D_1)_i\left(\varphi_i(x)-\varphi_i(x_0) - D\cdot(\varphi^i(x)-\varphi^i(x_0))\right).
\end{aligned}$$

Hence, because D is the derivative of φ_i at x_0 w.r.t. φ^i, we obtain that the second term also goes to zero. In conclusion, f is differentiable at x_0 w.r.t. φ^i, with derivative $(D_1)^i + (D_1)_iD$.

Exercise 5. Recall that (U, φ) is a chart if $\varphi\colon X \to \mathbb{R}^m$ is Lipschitz and every Lipschitz $f\colon X \to \mathbb{R}$ is differentiable with respect to φ a.e. in U.

Our first goal is to define the same for $f\colon X \to \mathbb{R}^n$. We define that f is differentiable at x_0 with respect to φ if there exists a unique n-by-m matrix D for which

$$\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - D(\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

The matrix D is then the differential of f at x_0.

Consider now the coordinate functions f_i of f, which are easily seen to be Lipschitz. For almost every $x_0 \in U$, we have differentials $D_i \in \mathbb{R}^m$ of the coordinate functions f_i at x_0 (this is just the definition of a chart, along with the obvious fact that the intersection of finitely many full measure sets is also of full measure). We now let D be the n-by-m matrix where the i:th row is D_i for all $i \in \{1, \dots, n\}$. Note that this yields the identity $Dv = (D_1 \cdot v, D_2 \cdot v, \dots, D_n \cdot v)$ for all $v \in \mathbb{R}^n$. Now we may just estimate

$$\begin{aligned}
&\limsup_{x \to x_0} \frac{|f(x) - f(x_0) - D(\varphi(x) - \varphi(x_0))|}{d(x, x_0)} \\
&= \limsup_{x \to x_0} \frac{\left| \sum_{i=1}^{n} \left(f_i(x) - f_i(x_0) - D_i \cdot (\varphi(x) - \varphi(x_0))\right) e_i \right|}{d(x, x_0)} \\
&\leq \sum_{i=1}^{n} \limsup_{x \to x_0} \frac{|f_i(x) - f_i(x_0) - D_i \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} \\
&= 0.
\end{aligned}$$

Hence, the differential D exists at every point $x_0 \in U$ where all coordinate functions of f are differentiable, which is almost everywhere in U.

Then, suppose D' is another differential at x_0, and let $D'_i \in \mathbb{R}^m$ be its rows. We have

$$\begin{aligned}
\left|f(x) - f(x_0) - D'(\varphi(x) - \varphi(x_0))\right| &\geq \left|(f(x) - f(x_0) - D'(\varphi(x) - \varphi(x_0)))_i\right| \\
&= \left|f_i(x) - f_i(x_0) - D'_i \cdot (\varphi(x) - \varphi(x_0))\right|.
\end{aligned}$$

Hence, D'_i is a differential of f_i at x_0. Since differentiability at x_0 for maps with 1-dimensional targets requires uniqueness of the differentials, we then have $D'_i = D_i$ if f_i is differentiable at x_0. Hence, $D' = D$ for almost every $x_0 \in U$. We now conclude that f is differentiable at almost every $x_0 \in U$.

Suppose now that $\varphi_1\colon X \to \mathbb{R}^{n_1}$ and $\varphi_2\colon X \to \mathbb{R}^{n_2}$ are two chart functions on U_1 and U_2, respectively, and suppose also that $\mu(U_1 \cap U_2) > 0$. By the above, φ_1 is differentiable almost everywhere in U_2 w.r.t. φ_2, and vice versa. Hence, since $\mu(U_1 \cap U_2) > 0$, we find a point $x_0 \in U_1 \cap U_2$ at which φ_1 is differentiable w.r.t. φ_2 and vice versa. Denote the differentials by D_1 and D_2, which are a n_1-by-n_2 and a n_2-by-n_1 matrix, respectively.

Suppose $n_1 > n_2$. Then the map $D_1 D_2 \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is not surjective, as the image of D_1 has dimension at most n_2. However, we may estimate

$$\begin{aligned}
&\limsup_{x \to x_0} \frac{|\varphi_1(x) - \varphi_1(x_0) - D_1 D_2(\varphi_1(x) - \varphi_1(x_0))|}{d(x, x_0)} \\
&\quad \le \limsup_{x \to x_0} \frac{|\varphi_1(x) - \varphi_1(x_0) - D_1(\varphi_2(x) - \varphi_2(x_0))|}{d(x, x_0)} \\
&\qquad + \limsup_{x \to x_0} \frac{|D_1\left(\varphi_2(x) - \varphi_2(x_0) - D_2(\varphi_1(x) - \varphi_1(x_0))\right)|}{d(x, x_0)} \\
&\quad \le \limsup_{x \to x_0} \frac{|\varphi_1(x) - \varphi_1(x_0) - D_1(\varphi_2(x) - \varphi_2(x_0))|}{d(x, x_0)} \\
&\qquad + \|D_1\| \limsup_{x \to x_0} \frac{|\varphi_2(x) - \varphi_2(x_0) - D_2(\varphi_1(x) - \varphi_1(x_0))|}{d(x, x_0)} \\
&\quad = 0,
\end{aligned}$$

where $\|D_1\|$ is the operator norm of D_1 (note that linear maps between finite-dimensional normed spaces have finite operator norms). Hence, φ_1 is has differential $D_1 D_2$ at x_0 w.r.t. itself. However, it is also clear that φ_1 has the differential id_{n_1} with respect to itself. Since differentials with respect to φ_1 are unique almost everywhere in U_1, we may assume x_0 is a point where this uniqueness holds, and therefore we have $D_1 D_2 = \mathrm{id}_{n_1}$. This is impossible, as $D_1 D_2$ is not surjective. A similar contradiction holds if $n_2 < n_1$. Hence, we must have $n_1 = n_2$.