# EXERCISE SHEET 3

**Exercise 1.** *For any metric space $X$ and any Lipschitz $f \colon X \to \mathbb{R}$ prove that $\rho(x) = \text{Lip}(f, x)$ defined in exercise sheet 2 is an upper gradient of $f$.*

**Exercise 2.** *For $\epsilon > 0$ let $D_\epsilon \subset \mathbb{R}^2$ be defined by*

$$D_\epsilon = S((-2, 0), 1) \cup S((2, 0), 1) \cup [-2, 2] \times [-\epsilon, \epsilon],$$

*for $S(x, r)$ the closed square centred at $x$ with side length $2r$. This set, with the Euclidean metric and Lebesgue measure, is a 1-PI space (you do not have to prove this). Prove that the constant appearing in the Poincaré inequality converges to infinity as $\epsilon \to 0$. Hint: consider the function*

$$f((x, y)) = \begin{cases} -1 & (x, y) \in S((-2, 0), 1) \\ 1 & (x, y) \in S((2, 0), 1) \\ x & otherwise. \end{cases}$$

*This example shows that, although this domain has plenty of curves, the "bottle neck" in the centre forces a large constant in the Poincaré inequality. Thus, if a space satisfies a Poincaré inequality, it is impossible to find subsets with worse and worse bottle necks.*

**Exercise 3.** *Let $(X, d, \mu)$ be a metric measure space and $(Y, d')$ a metric space. Suppose that $\iota \colon X \to Y$ is biLipschitz and surjective and define $\nu = f_{\#}\mu$ (the push-forward of $\mu$).*

*Suppose that $(X, d, \mu)$ is doubling. Show that $(Y, d', \nu)$ is also doubling.*

*Suppose that $(X, d, \mu)$ satisfies a 1-Poincaré inequality and is doubling. Show that $(Y, d', \nu)$ satisfies a weak 1-Poincaré inequality: there exist $C, \lambda \geq 1$ such that, for every Lipschitz $f \colon Y \to \mathbb{R}$, every upper gradient $\rho$ of $f$ and every ball $B \subset Y$,*

$$\fint_B |f - f_B| d\mu \leq C \, \text{rad}(B) \fint_{\lambda B} \rho \, d\mu.$$

*Note that the only difference to a (strong) Poincaré inequality is the $\lambda$ in the domain of the second integral.*

**Exercise 4.** *Recall that in the definition of a derivative with respect to a chart function $\phi \colon X \to \mathbb{R}^n$ we include the requirement that the derivative is unique. Suppose for this question that we do not.*

   i) *Prove that $\mathbb{R}$ with the chart map $\phi(x) = (x, x) \in \mathbb{R}^2$ is a Lipschitz differentiability space.*

*This phenomenon that a non-unique derivative arises whenever one of the components of the chart map is differentiable with respect to the others is typical; like reducing a spanning set of vectors to a linearly independent set, we can always reduce from the case of a non-unique derivative to a unique derivative.*

*From now on, suppose that a Lipschitz $f \colon X \to \mathbb{R}$ has two derivatives $D_1$ and $D_2$ at $x_0 \in X$ with respect to the chart map $\phi \colon X \to \mathbb{R}^n$.*

   ii) *Prove that*

$$\limsup_{x \to x_0} \frac{|(\phi(x) - \phi(x_0)) \cdot (D_1 - D_2)|}{d(x, x_0)} = 0.$$

iii) *Deduce that, for some choice of $i \in \{1, 2, \ldots, n\}$, $\phi_i$ is differentiable at $x_0$ with respect to the chart map $\phi^i \colon X \to \mathbb{R}^{n-1}$ which excludes the $i^{th}$ component.*

iv) *Deduce that, for this choice of $i$, $f$ is differentiable at $x_0$ with respect to $\phi^i$.*

*Thus, if $X$ is a Lipschitz differentiability space with respect to $\phi$, by defining $U_i$ to be those $x_0 \in X$ for which $i$ was chosen, we obtain a new collection of charts, $(U_1, \phi^1), (U_2, \phi_2), \ldots, (U_n, \phi^n)$, with respect to which $X$ is also a Lipschitz differentiability space (and the dimension of each chart is strictly smaller). We can continue this inductively, eventually ending up at a unique derivative with respect to some chart of dimension $\geq 0$.*

**Exercise 5.** *Let $(X, d, \mu)$ be a Lipschitz differentiability space and $(U, \phi)$ a chart. Suppose that $f \colon X \to \mathbb{R}^n$ is Lipschitz. Write down a definition for what it means for $f$ to be differentiable at a point $x_0 \in U$ with respect to $\phi$. Prove that $f$ is differentiable $\mu$ almost everywhere in $U$.*

*For $\phi_1 \colon X \to \mathbb{R}^{n_1}$ and $\phi_2 \colon X \to \mathbb{R}^{n_2}$, suppose that $(U_1, \phi_1)$ and $(U_2, \phi_2)$ are two charts such that $\mu(U_1 \cap U_2) > 0$. Prove that $n_1 = n_2$. Hint: What happens at a point where $\phi_1$ is differentiable with respect to $\phi_2$ and where $\phi_2$ is differentiable at $\phi_1$? Why must such a point exist?*