EXERCISE SHEET 3

Exercise 1. *For any metric space X and any Lipschitz $f\colon X \to \mathbb{R}$ prove that $\rho(x) = \mathrm{Lip}(f, x)$ defined in exercise sheet 2 is an upper gradient of f.*

Exercise 2. *For $\epsilon > 0$ let $D_\epsilon \subset \mathbb{R}^2$ be defined by*

$$D_\epsilon = S((-2,0),1) \cup S((2,0),1) \cup [-2,2] \times [-\epsilon, \epsilon],$$

for $S(x,r)$ the closed square centred at x with side length $2r$. This set, with the Euclidean metric and Lebesgue measure, is a 1-PI space (you do not have to prove this). Prove that the constant appearing in the Poincaré inequality converges to infinity as $\epsilon \to 0$. Hint: consider the function

$$f((x,y)) = \begin{cases} -1 & (x,y) \in S((-2,0),1) \\ 1 & (x,y) \in S((2,0),1) \\ x & \text{otherwise.} \end{cases}$$

This example shows that, although this domain has plenty of curves, the "bottle neck" in the centre forces a large constant in the Poincaré inequality. Thus, if a space satisfies a Poincaré inequality, it is impossible to find subsets with worse and worse bottle necks.

Exercise 3. *Let (X, d, μ) be a metric measure space and (Y, d') a metric space. Suppose that $\iota\colon X \to Y$ is biLipschitz and surjective and define $\nu = f_{\#}\mu$ (the push-forward of μ).*

Suppose that (X, d, μ) is doubling. Show that (Y, d', ν) is also doubling.

Suppose that (X, d, μ) satisfies a 1-Poincaré inequality and is doubling. Show that (Y, d', ν) satisfies a weak *1-Poincaré inequality: there exist $C, \lambda \geq 1$ such that, for every Lipschitz $f\colon Y \to \mathbb{R}$, every upper gradient ρ of f and every ball $B \subset Y$,*

$$⨍_B |f - f_B| d\mu \leq C \operatorname{rad}(B) ⨍_{\lambda B} \rho d\mu.$$

Note that the only difference to a (strong) Poincaré inequality is the λ in the domain of the second integral.

Exercise 4. *Recall that in the definition of a derivative with respect to a chart function $\phi\colon X \to \mathbb{R}^n$ we include the requirement that the derivative is unique. Suppose for this question that we do not.*

i) *Prove that $\mathbb{R}$ with the chart map $\phi(x) = (x,x) \in \mathbb{R}^2$ is a Lipschitz differentiability space.*

This phenomenon that a non-unique derivative arises whenever one of the components of the chart map is differentiable with respect to the others is typical; like reducing a spanning set of vectors to a linearly independent set, we can always reduce from the case of a non-unique derivative to a unique derivative.

From now on, suppose that a Lipschitz $f\colon X \to \mathbb{R}$ has two derivatives D_1 and D_2 at $x_0 \in X$ with respect to the chart map $\phi\colon X \to \mathbb{R}^n$.

ii) *Prove that*

$$\limsup_{x \to x_0} \frac{|(\phi(x) - \phi(x_0)) \cdot (D_1 - D_2)|}{d(x, x_0)} = 0.$$

iii) *Deduce that, for some choice of* $i \in \{1, 2, \ldots, n\}$, ϕ_i *is differentiable at* x_0 *with respect to the chart map* $\phi^i \colon X \to \mathbb{R}^{n-1}$ *which excludes the* i^{th} *component.*

iv) *Deduce that, for this choice of* i, f *is differentiable at* x_0 *with respect to* ϕ^i.

Thus, if X *is a Lipschitz differentiability space with respect to* ϕ, *by defining* U_i *to be those* $x_0 \in X$ *for which* i *was chosen, we obtain a new collection of charts,* $(U_1, \phi^1), (U_2, \phi_2), \ldots, (U_n, \phi^n)$, *with respect to which* X *is also a Lipschitz differentiability space (and the dimension of each chart is strictly smaller). We can continue this inductively, eventually ending up at a unique derivative with respect to some chart of dimension* ≥ 0.

Exercise 5. *Let* (X, d, μ) *be a Lipschitz differentiability space and* (U, ϕ) *a chart. Suppose that* $f \colon X \to \mathbb{R}^n$ *is Lipschitz. Write down a definition for what it means for* f *to be differentiable at a point* $x_0 \in U$ *with respect to* ϕ. *Prove that* f *is differentiable* μ *almost everywhere in* U.

For $\phi_1 \colon X \to \mathbb{R}^{n_1}$ *and* $\phi_2 \colon X \to \mathbb{R}^{n_2}$, *suppose that* (U_1, ϕ_1) *and* (U_2, ϕ_2) *are two charts such that* $\mu(U_1 \cap U_2) > 0$. *Prove that* $n_1 = n_2$. *Hint: What happens at a point where* ϕ_1 *is differentiable with respect to* ϕ_2 *and where* ϕ_2 *is differentiable at* ϕ_1*? Why must such a point exist?*