

EXERCISE SHEET 4

Exercise 1. A previous exercise asked you to prove that a Poincaré inequality is preserved under bi-Lipschitz transformations. I cannot find an easy way to prove this. However, we do not need it for the proof of Cheeger's theorem. We only require the following.

Let (X, d, μ) be a metric measure space for which there exists $C, \eta \geq 1$ such that, for any 1-Lipschitz $f: X \rightarrow \mathbb{R}$ and any upper gradient ρ of f ,

$$\text{Lip}(f, x)^\eta \leq C\rho(x)$$

for μ almost every $x \in X$.

Suppose that $\iota: X \rightarrow (Y, d')$ is bi-Lipschitz and $\nu = \iota_\# \mu$. Show that there exist $C', \eta' \geq 1$ such that, for any 1-Lipschitz $f: Y \rightarrow \mathbb{R}$ and any upper gradient ρ of f ,

$$\text{Lip}(f, y)^{\eta'} \leq C'\rho(y)$$

for ν almost every $y \in Y$.

Exercise 2. Prove the converse to Proposition 6.2: Suppose that (U, ϕ) is a chart in a Lipschitz differentiability space. Show that there exists an $N \in \mathbb{N}$ such that, whenever $\psi: X \rightarrow \mathbb{R}^n$ is a Lipschitz function with

$$\text{Lip}(D \cdot \psi, x) > 0$$

for every $x \in U$ and every $D \in \mathbb{R}^n \setminus \{0\}$, then $n \leq N$.

Exercise 3. For a metric space (X, d) , let \mathcal{C} be the collection of non-empty closed and bounded subsets of X . For $C, D \in \mathcal{C}$ define the Hausdorff distance between C and D to be

$$d_H(C, D) = \inf\{r \geq 0 : C \subset B(D, r) \text{ and } D \subset B(C, r)\},$$

for $B(D, r)$ the r -neighbourhood of D :

$$B(D, r) = \bigcup_{d \in D} B(d, r).$$

- i) Prove that d_H is a metric on \mathcal{C} .
- ii) Give an example to show that d_H may not be a metric on the set of non-empty bounded subsets of a metric space.

Exercise 4. Show that, if X is complete, then so is \mathcal{C} . Specifically, let C_n be a Cauchy sequence in \mathcal{C} . By taking a subsequence, we may suppose that $d(C_n, C_m) < 2^{-n}$ for all $m \geq n \in \mathbb{N}$. Define C to be the set

$$C := \{c \in X : \forall n \in \mathbb{N} \exists c_n \in C_n, c_n \rightarrow c\}.$$

- i) Show that C is non-empty, bounded and closed (and hence belongs to \mathcal{C}).
- ii) Given $\epsilon > 0$ show that there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $C \subset B(C_n, \epsilon)$.
- iii) For the other inclusion, for any $N \in \mathbb{N}$ and any $c^* \in C_N$ construct a Cauchy sequence $c_n \in C_n$ for which c_n converges to some point $c \in C$ with $d(c, c^*) \leq 2^{-N+1}$. In particular, $C_j \subset B(C, 2^{-N+1})$. Therefore, $C_n \rightarrow C$.

Exercise 5. Show that, if X is totally bounded, then so is \mathcal{C} . Hint: for any $\epsilon > 0$ let

$$X \subset B(x_1, \epsilon) \cup \dots \cup B(x_m, \epsilon)$$

and consider sets of the form

$$\bigcup_{i=1}^j B(y_i, \epsilon)$$

with $j \leq m$ and each $y_i \in \{x_1, \dots, x_m\}$.

Therefore, if X is compact, so is \mathcal{C} . This is known as Blaschke's theorem.

Exercise 6. There is another proof of the compactness statement. In fact, the compactness of \mathcal{C} is equivalent to the Ascoli–Arzelà theorem.

Suppose that X is compact and $C_n \in \mathcal{C}$.

- i) Prove that the functions $f_n(x) = d(x, C_n)$ are all 1-Lipschitz and there exists a closed interval $[0, d]$ into which each f_n is defined. Therefore, by the Ascoli–Arzelà theorem, (after possibly taking a subsequence) there exists a 1-Lipschitz $f: X \rightarrow \mathbb{R}_+$ such that $f_n \rightarrow f$.
- ii) Show that

$$C_n \rightarrow C := \{x \in X : f(x) = 0\}.$$

Now suppose that $f_n: X \rightarrow Y$ is a sequence of L -Lipschitz functions between two compact metric spaces. For each $n \in \mathbb{N}$ define C_n to be the graph of f_n :

$$C_n = \{(x, f_n(x)) \in X \times Y : x \in X\}.$$

- iii) Show that each C_n is a compact subset of $X \times Y$.
- iv) If we assume that $\mathcal{C}(X \times Y)$ is compact, then (after passing to a subsequence) there exists a $C \in \mathcal{C}$ with $C_n \rightarrow C$. Show that C is the graph of a function $f: X \rightarrow Y$.
- v) Prove that $f_n \rightarrow f$ uniformly.